GATE 2010 – SET 1

COMPUTER SCIENCE AND INFORMATION TECHNOLOGY (CS)

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There is no magic formula for success.
Just 3 essential ingredients which,
when combined, can work wonders.

The desire to succeed

The willingness to work hard

The right preparatory guidance

BRILLIANT TUTORIALS PRIVATE LIMITED

GATE

SET 1

COMPUTER SCIENCE AND INFORMATION TECHNOLOGY (CS)

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SET 1

LETTER TO THE STUDENT

Dear Student,

We welcome you to our comprehensive postal coaching programme for GATE 2010. We are glad that you have thus considerably enhanced your prospects in GATE.

Our course material is based on the syllabus for GATE 2009. The syllabus and topics planned to be covered in each Set of course material and the structure of GATE 2009 are given in this Set for your reference. The syllabus for GATE 2010 will be announced only by Mid-September 2009. If there is a change in the syllabus, we shall provide supplements to cover the change as far as possible. The structure of GATE 2009 question paper was of "All Objective" type with 20 one mark questions and 40 two marks questions with 1/3rd negative mark for wrong answers.

Our course material includes Assignments, GATE questions for a few previous years and four Model Test Papers on the pattern of GATE. We strongly advise you to attempt and answer the questions on your own, before referring to the answers provided by us. Only such practice will boost your confidence to face the examination.

Our new student-centric initiative of 'On-line' tests got encouraging response for GATE 2009. The experience and practise of answering the question paper on-line, under strict simulated examination conditions is sure to boost your skill and preparedness for GATE 2010. We suggest and strongly recommend that you also opt for the same, if you have not done so far.

Our Doubt Letter Scheme is a facility for you to clarify doubts and to get your answers evaluated. You should make full use of it, strictly following the guidelines given.

GATE examination is highly competitive. Take it as a challenge, plan and study hard, and practice answering questions setting yourself a time limit.

We wish you success in GATE and in your future career.

With best wishes,

Yours truly,
For BRILLIANT TUTORIALS PVT. LTD.,

Coordinator
(GATE/ESE/UGC)

GENERAL INSTRUCTIONS

Our Course Material

Our course material for the subject will be in 8 Sets. Each Set has in general five sections viz., Lessons on Theory as per the syllabus, Worked Examples, Objective Type Questions with key and explanatory notes wherever necessary, Assignments and Solutions to Assignments.

While the emphasis is on objective type questions, utilize our material in its entirety, as that will be the right approach to tackle the pattern of all-objective type questions.

Our study materials are not substitutes for text books, but they form an effective supplement to the Standard Text Books, to widen and enhance your knowledge and preparation for GATE.

All-objective Pattern of Test

The pattern of Questions for GATE 2008 and 2009 was of all-objective type. GATE 2010 is also expected to follow the same pattern. All questions carry marks. Answer all the questions. Your score entirely depends on fast and correct response.

Selective Preparation/Omission of Topics

Generally the paper covers the entire syllabus in a balanced manner. **Selective reading or missing of topics based on predictions may be risky and put you at a disadvantage.**

Time Budgeting

180 minutes are available to answer 60 questions, which includes reading the text with the choices, mentally working out, deciding the choice and marking. The average time available is 3 minutes per question. Some questions, which are not so lengthy, can be read and understood in 30 to 40 seconds. Others which are lengthy may take 40 to 60 seconds, to read and re-read, before deciding. **You must keep an eye on the watch during the examination.**

Spending more time on a question and attempting it again and again will upset your time budget. **Skip seemingly difficult and complex questions during the first reading.**

In the second round, attempt the unanswered questions, by eliminating the clearly wrong options and spotting the correct answers from the remaining options.

Tough Paper

If the paper **seems tough on the first reading,** do not get nervous. You stand as good a chance, as the others.

Same answer key for many questions

Do not get disturbed if you get the same answer continuously for many questions. It is a common feature in all-objective competitive type examinations.

Worked examples and Assignments

Objective questions can be various types as listed in structure of GATE. Common data questions and linked answer questions are only different forms of worked examples. Our sets contain a large number of worked examples. A study of and answering these in fact, will equip you with sound theoretical knowledge and preparedness for answering all the types of objective questions.

Model Test Papers and Questions and Answers of Previous Years

You will find four Model Test Papers and Questions and Answers of last five years of GATE, in different Sets. Answer the questions, on your own, setting yourself a time limit and without referring to the answers given in the sets. This will give you the best possible practice. Since 2004, the pattern of questions is of all-objective type. Question Papers earlier to 2004 contain descriptive questions and answers. These will facilitate understanding the theory and formulae, answering common data and linked answer questions.

Doubt Letter Scheme

Our Professors services are readily available to clear your doubts and difficulties while studying our course material.

Our Professors are hard pressed for time. So there must be reasonable restrictions on reference made by the students. Please follow the guidelines given below for Doubt Letters.

Do not send Doubt Letters on postcards.

Refer only five problems at a time, provided these have been first attempted by you. The working sheets giving your approach and the difficulty encountered must be sent with the Doubt Letter. Only then the Professor can guide you properly. The source of the problems should always be mentioned. Do not copy questions from a book and forward.

Write down the problems clearly and accurately. In case of doubts pertaining to our course material, **quote the Set Number, Page Number, Assignment Number and Problems Number**.

Avoid mixing different subjects under XL on the same sheet of paper of Doubt Letter.

You can refer your doubts to us up to the second week of December 2009.

You are bound to gain a lot, if you utilize the Doubt Letter Scheme, systematically, strictly following the above guidelines.

STRUCTURE OF GATE 2009 – EXTRACTS

The GATE 2009 examination consists of a **single paper of 3 hours duration and of 100 marks.**

The choice of the appropriate paper is the responsibility of the candidate. However, some guidelines are suggested below:

(a) **Candidates are expected to appear in a paper appropriate to the discipline of their qualifying degree.**

(b) **However, the candidates are free to choose any paper according to their admission plan, keeping in mind the eligibility criteria of the admitting institute.**

The question paper of GATE 2009 will be fully of objective type.

(a) Candidates have to mark the correct choice by **darkening** the appropriate **bubble** against each question on an **Objective Response Sheet (ORS).**

(b) There will be **'negative' marking** for wrong answers. The deduction will be **1/3rd** of the marks allotted.

Main Papers (Except XE and XL)

(a) The question paper will be for a total of 100 marks divided into two groups.

(i) **Group I:** Question Numbers 1 to 20 (20 questions) will carry one mark each (subtotal 20 marks).

(ii) **Group II:** Question Numbers 21 to 60 (40 questions) will carry two marks each (subtotal 80 marks). Out of this Q.51 to Q.56 will be common data based questions. Question Pairs 57, 58 and 59, 60 (4 questions) will carry two marks each. These questions are called linked answer questions. These 4 questions comprise two pairs of questions. The solution to the second question of each pair (e.g. Q.58) will be linked to the correct answer to the first one (e.g. Q.57) in the pair.

(b) Each question will have four choices for the answer. **Only one** choice is correct.

(c) In Q.1 to Q.20, **1/3rd** mark will be deducted for each wrong answer and in Q.21 to Q.56, Q.57 and Q.59, **2/3rd** mark will be deducted for each wrong answer. However, there is no negative marking in Q.58 and Q.60.

(d) Papers bearing the code AG, CE, CS, CH, EC, EE, IN, IT, ME, MN, MT, PI, TF will contain a few questions on Engineering Mathematics.

(e) The multiple choice objective test questions can be of the following type:

Multiple choice questions in all papers and sections will contain four answers, of which only one is correct. The type of questions in a paper may be based on following logic:

(i) **Recall**

These are based on facts, principles, formulae or laws of the discipline. The candidate is expected to be able to obtain the answer either from his/her memory of the subject or at most from a one-line computation.

(ii) **Comprehension**

These questions will test the candidate's understanding of the basics of his/her field, by requiring him/her to draw simple conclusions from fundamental ideas.

(iii) Application

In these questions, the candidate is expected to apply his/her knowledge either through computation or by logical reasoning.

(iv) Analysis and Synthesis

These can be linked questions, where the answer to the first question of the pair is required in order to answer its successor. Or these can be common data questions, in which two questions share the same data but can be solved independently of one another.

The questions based on the above four logics may be a mix of single stand alone statement/phrase/data type questions, combination of option codes type questions or match items type questions.

Common data questions

Multiple questions may be linked to a common data problem, passage and the like. Two or three questions can be formed from the given common data problem. Each question is independent and its solution obtainable from the above problem data/passage directly. (Answer of the previous question is not required to solve the next question.) Each question under this group will carry two marks.

Linked answer questions

These questions are of problem solving type. A problem statement is followed by two questions based on the problem statement. The two questions are designed such that the solution to the second question depends upon the answer to the first one. In other words, the first answer is an intermediate step in working out the second answer. Each question in such 'linked answer questions' will carry two marks.

SYLLABUS – GATE 2009

COMPUTER SCIENCE AND INFORMATION TECHNOLOGY (CS)

ENGINEERING MATHEMATICS

Mathematical Logic: Propositional Logic; First Order Logic.

Probability: Conditional Probability; Mean, Median, Mode and Standard Deviation; Random Variables; Distributions; uniform, normal, exponential, Poisson, Binomial.

Set Theory & Algebra: Sets; Relations; Functions; Groups; Partial Orders; Lattice; Boolean Algebra.

Combinatorics: Permutations; Combinations; Counting; Summation; generating functions; recurrence relations; asymptotics.

Graph Theory: Connectivity; spanning trees; Cut vertices & edges; covering; matching; independent sets; Colouring; Planarity; Isomorphism.

Linear Algebra: Algebra of matrices, determinants, systems of linear equations, Eigenvalues and Eigenvectors.

Numerical Methods: LU decomposition for systems of linear equations; numerical solutions of non-linear algebraic equations by Secant, Bisection and Newton-Raphson Methods; Numerical integration by trapezoidal and Simpson's rules.

Calculus: Limit, Continuity & differentiability, Mean value Therorems, Theorems of integral calculus, evaluation of definite & improper integrals, Partial derivatives, Total derivatives, maxima & minima.

COMPUTER SCIENCE AND INFORMATION TECHNOLOGY

Digital Logic: Logic functions, Minimization, Design and synthesis of combinational and sequential circuits; Number representation and computer arithmetic (fixed and floating point).

Computer Organization and Architecture: Machine instructions and addressing modes, ALU and data-path, CPU control design, Memory interface, I/O interface (Interrupt and DMA mode), Instruction pipelining, Cache and main memory, Secondary storage.

Programming and Data Structures: Programming in C; Functions, Recursion, Parameter passing, Scope, Binding; Abstract data types, Arrays, Stacks, Queues, Linked Lists, Trees, Binary search trees, Binary heaps.

Algorithms: Analysis, Asymptotic notation, Notions of space and time complexity, Worst and average case analysis; Design: Greedy approach, Dynamic programming, Divide-and-conquer; Tree and graph traversals, Connected components, Spanning trees, Shortest paths; Hashing, Sorting, Searching. Asymptotic analysis (best, worst, average cases) of time and space, upper and lower bounds, Basic concepts of complexity classes –P, NP NP-hard, NP-complete.

Theory of Computation: Regular languages and finite automata, Context free languages and Push-down automata, Recursively enumerable sets and Turing machines, Undecidability

Compiler Design: Lexical analysis, Parsing, Syntax directed translation, Runtime environments, Intermediate and target code generation, Basics of code optimization.

Operating System: Processes, Threads, Inter-process communication, Concurrency, Synchronization, Deadlock, CPU scheduling, Memory management and virtual memory, File systems, I/O systems, Protection and security.

Databases: ER-model, Relational model (relational algebra, tuple calculus), Database design (integrity constraints, normal forms), Query languages (SQL), File structures (sequential files, indexing, B and B+ trees), Transactions and concurrency control.

Information Systems and Software Engineering: Information gathering, requirement and feasibility analysis, data flow diagrams, process specifications, input/output design, process life cycle, planning and managing the project, design, coding, testing, implementation, maintenance.

Computer Networks: ISO/OSI stack, LAN technologies (Ethernet, Token ring), Flow and error control techniques, Routing algorithms, Congestion control, TCP/UDP and sockets, IP (v4), Application layer protocols (icmp, dns, smtp, pop, ftp, http); Basic concepts of hubs, switches, gateways and routers. Network security – basic concepts of public key and private key cryptography, digital signature, firewalls.

Web technologies: HTML, XML, basic concepts of client-server computing.

TOPICS PROPOSED TO BE COVERED IN EACH SET

Set No.	Topic
1	Engineering Mathematics
2	Engineering Mathematics
3	Computer Hardware – Digital Logic, Computer Organization
4	Theory of Computation
5	Software Systems – Data Structure
6	Operating Systems
7	Data Bases, including Information Systems and Software Engineering
8	Computer Networks, including HTML, Network security, Client server computing and GATE Questions and Answers.

21.	Ian Anderson	Combinatorics of Finite sets	Dover Publications
22.	Peter J. Cameron	Combinatorics: Topics, Techniques, Algorithms	Cambridge University Press
23.	John Riordan	Introduction to Combinatorial analysis	Dover Publications
24.	Gray Chartrand	Introductory Graph Theory	Dover Publications
25.	J.A. Bondy and U.S.R. Murthy	Graph Theory with Applications	MacMillan
26.	H.C.S. Saxena	Numerical Analysis	S. Chand & CO.
27.	B.D. Gupta	Numerical Analysis	Stosivs inc/Advent Books Division
28.	V. Rajaraman	Computer Oriented Numerical Methods	Prentice Hall of India
29.	Peter Linz	An Introduction to Formal Languages and Automata	Jones and Bartlett Publications
30.	J. Glenn Brookshear	Theory of Computation; Formal Languages, Automata and Complexity	The Benjamin/Cummings Publishing Company Inc.
31.	Mano, M. Morris	Digital Logic and Computer Design.	Prentice Hall
32.	Andrew. S. Tanenbaum.	Computer Organisation	Prentice Hall
33.	David A. Patterson, John. L. Hennessy	Computer Organisation and Design	Morgan Kaufman
34.	Alfred V. Aho, Jeffrer. D. Ullman, John E Hopcroft	Data Structures and Algorithms in Computer Science and information	Morgan Kaufman
35.	Kerninghan and Ritchie	The C-Programming Languages	Prentice Hall
36.	E. Balagurusamy	Programming in ANSI C.	Tata-McGraw-Hill
37.	Robert Lafore	Object oriented programming	Waibe Group Press
38.	Alfred V. Aho and Jeffrey D. Ullman	Principle of Compiler Design	Addison Wesley
39.	Andrew Tanenbaum	Modern Operating System	Pearson Education
40.	Andrew S. Tanenbaum	Distributed operating system	Prentice Hall, Inc.
41.	Silberschatz Galvin	Operating system Concepts	Prentice Hall, Inc.

42.	Bach	The Design of Unix OS	Prentice Hall
43.	Roger	Software Engineering-A practitioner's Approach	McGraw-Hill
44.	Guezzi, Jayazeni and Mandrioli	Fundamentals of Software Engineering	
45.	Ian Sommerville	Software Engineering	Addison Wesley
46.	Schach	Software Engineering	A.K. Sen Associates
47.	C.J. Date	Introduction to Database systems	Addison Wesley Longman
48.	Korth and Silberschatz	Database system concepts	McGraw-Hill
49.	Elmasri/Navathe	Fundamentals of Database systems	Benjamin-Cummings
50.	Andrew Tanenbaum	Computer Networks	Prentice Hall
51.	Leon. Garcia, Widjaja	Communication Network, Fundamental Concepts and Key Architectures	
52.	Peterson and Davies	Computer Networks, Systems Approach	Morgan-Kaufman
53.	Teach You SQL		BPB Publications
54.	Mastering SQL		BPB Publications
55.	Steven Holzner	HTML Black Book	Dream Tech Press
56.	James Jaworski	Java Script and J script	Sybex. Inc.
57.	Andrea Steelman Joel Murach	Java Servlets and J.S.P	Murach Publications
58.	Alex Homer, Dave Sussman	ASP NET Professional	
59.	Erik. T. Ray (Spider Pro)	Learning XML	
60.	David Hunser, Jeff Rafter	Beginning XML	

Website References

http://www.ericweisstein.com/encyclopedias/books/discrete mathematics.html.

http://www.graphtheory.com

http://www.numerical_methods.com

http://www.math.jet.ac.illnnaiman/nm

1. MATHEMATICAL LOGIC

PROPOSITIONAL LOGIC

Introduction

A **proposition** is a declarative statement that is either **true** or **false** but not both. For example, "My name is Ramesh" is a declarative statement but "what is your name?" is not a declarative statement.

Note that when Ramesh tells "My name is Ramesh", it is true and when Suresh tells the same statement, it is false. Thus a proposition is a statement involving only definite or constant notions. The truthfulness or falsity of a proposition is called its **truth value**; when the proposition is true, its truth value is T and when false, it is F. Propositions formed out of **subpropositions** using **connectives** are called **compound propositions**. The basic property of a compound proposition is that its truth value is decided by the truth values of the subpropositions in it along with the manner in which they are linked to form the compound proposition.

Definition of Tautology

A compound proposition that is always true no matter what the truth values of the propositions that occur in it, is called a **tautology**.

A compound proposition that is always false irrespective of the truth values of the constituent propositions is called a **contradiction**.

A proposition that is neither a tautology nor a contradiction is called a **contingency**.

Truth Table: A truth table displays the relationships between the truth values of propositions.

1. **Atomic statement:** Statements without connectives like 'but', 'or', 'and' are called **atomic** or **primary statements**.
2. **Conjunction:** Let P and Q be two statements. Then the statement "P and Q" denoted by $P \wedge Q$ is called the **conjunction** of P and Q.

   Let P : I am Ramesh

   Q : He is Suresh

   Then $P \wedge Q$: I am Ramesh and he is Suresh.
3. **Disjunction:** $P \vee Q$ meaning "P or Q" is called the **disjunction** of P and Q.

   For the above statements P and Q, $P \vee Q$ is "I am Ramesh or he is Suresh".
4. **Negation:** 'Not P', denoted by $\rceil P$ or by $\sim P$ is called the **negation** of P.

   For the above example, $\rceil P$ is "I am not Ramesh".

   The truth table for $\vee$, $\wedge$ and $\rceil$ is given in the following table.

P	Q	$P \vee Q$	$P \wedge Q$	$\rceil P$
T	T	T	T	F
T	F	T	F	F
F	T	T	F	T
F	F	F	F	T

5. **Conditional:** The statement "If P then Q" is called the **conditional statement** and it is denoted by $P \rightarrow Q$. In $P \rightarrow Q$, P is called the **antecedent** and Q is the **consequent.** $P \rightarrow Q$ is false only when P is true and Q is false. For all other alternatives, $P \rightarrow Q$ is always true. A true statement cannot have a false implication.
6. **Biconditional:** The statement "P if and only if Q", denoted by $P \rightleftarrows Q$ is called the **biconditional statement**. In this situation, we say that Q is a necessary and sufficient condition for P. The biconditional $P \rightleftarrows Q$ is true only when P and Q have the same truth values. $P \rightleftarrows Q$ is used to mean $P \rightarrow Q$ and $Q \rightarrow P$. We say that 'P is equivalent to Q'.

   The **truth table** for $P \rightarrow Q$ and $P \rightleftarrows Q$ is given in the following **table.**

P	Q	$P \rightarrow Q$	$P \rightleftarrows Q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

Remark 1: The symbols $\rceil$, $\vee$, $\wedge$, $\rightarrow$, $\rightleftarrows$ are called **logical constants**; the last four of these, which are used between two propositions, are called **logical connectives.**

The following **table** defines these logical operators:

Let P, Q be any two propositions.

Terminology	Notation	Description
Negation of P	$\urcorner P$	not P
Disjunction of P, Q	$P \vee Q$	P or Q
Conjunction of P, Q	$P \wedge Q$	P and Q
Implication or conditional	$P \rightarrow Q$	If P then Q
Bi-implication or Biconditional	$P \rightleftarrows Q$	Q if and only if P

It is customary to use the **abbreviation iff for 'if and only if'**. All definitions are iff statements.

Remark 2: Exclusive OR: Let P and Q be propositions. Then $P \bar{\vee} Q$ means either P or Q but never both. Exclusive OR is also denoted by $P \oplus Q$.

Remark 3: By $P \equiv T$ we mean that P is true. $Q \equiv F$ means that Q is false.

7. Statement formula: A string of symbols is said to form a **well-formed statement formula** (abbreviated as WFF), if it is derived by applying the following rules:-

(i) Any statement P is a WFF

(ii) If A is a WFF then $\urcorner A$ is WFF

(iii) If A and B are WFFs then $A \wedge B$, $A \vee B$, $A \rightarrow B$ and $A \rightleftarrows B$ are WFFs.

For example, $(P \rightarrow Q) \vee (R \rightleftarrows S)$ is a WFF but $(\urcorner(P \rightarrow Q) \wedge)$ is not.

8. Logical Equivalence: Two propositions P and Q are said to be **logically equivalent or equivalent** or simply **equal**, denoted by $P \equiv Q$ if they have identical truth tables.

The following truth **tables** show that $\urcorner(P \wedge Q) \equiv \urcorner P \vee \urcorner Q$

P	Q	$P \wedge Q$	$\urcorner(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

$\urcorner(P \wedge Q)$

P	Q	$\urcorner P$	$\urcorner Q$	$\urcorner P \vee \urcorner Q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$\urcorner P \vee \urcorner Q$

9. Implication: A formula P is said to imply Q, if $P \rightarrow Q$ is true and we write then $P \Rightarrow Q$.

10. Equivalent Formulae: Two formulae P and Q are equivalent, denoted by $P \Leftrightarrow Q$, iff $P \rightleftarrows Q$ is true.

11. NAND: $P \uparrow Q \Leftrightarrow \urcorner(P \wedge Q)$

12. NOR: $P \downarrow Q \rightleftarrows \urcorner(P \vee Q)$

Thus NAND and NOR stand for Not AND and Not OR respectively.

The truth table for $\bar{\vee}$, $\uparrow$ and $\downarrow$ is given in Table below.

P	Q	$P \bar{\vee} Q$	$P \uparrow Q$	$P \downarrow Q$
T	T	F	F	F
T	F	T	T	F
F	T	T	T	F
F	F	F	T	T

We list the laws of algebra of propositions as under. t, f, $\equiv$ stand for tautology, contradiction and logical equivalence.

	Laws	Algebra of Propositions
1.	Idempotent laws	$P \vee P \equiv P$; $P \wedge P \equiv P$
2.	Associative laws	$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$ $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
3.	Commutative laws	$P \vee Q \equiv Q \vee P$; $P \wedge Q \equiv Q \wedge P$
4.	Distributive laws	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
5.	Identity laws	$P \vee f \equiv P$, $P \wedge t \equiv P$ $P \vee t \equiv t$, $P \wedge f \equiv f$
6.	Complement laws	$P \vee \urcorner P \equiv t$, $P \wedge \urcorner P \equiv f$ $\urcorner\urcorner P \equiv P$, $\urcorner t \equiv f$, $\urcorner f \equiv t$

Laws	Algebra of Propositions
7. De Morgan's laws	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
8. Absorption laws	$P \vee (P \wedge Q) \equiv P$, $P \wedge (P \vee Q) \equiv P$.

13. Functionally complete set: A subset of connectives ($\wedge, \vee, \neg, \longrightarrow, \rightleftarrows$) is said to be **functionally complete,** if any formula can be converted into an equivalent formula, having only the connectives from the above subset.

For example, $\{\neg, \wedge\}$ is a functionally complete set, but $\{\neg\}$ and $\{\wedge, \vee\}$ are not.

14. Normal Forms: The normal form of a given propositional formula is devoid of the connectives $\longrightarrow$ and $\rightleftarrows$. The connectives $\vee$, $\wedge$ (called sum and product respectively) and $\neg$ alone are present in such normal forms. Thus the given formula can be reduced to a **conjunctive normal form** or a **disjunctive normal form.**

Examples

(a) **Product:** $P \wedge Q$, $\neg P \wedge Q \wedge R$

(b) **Sum:** $P \vee Q$, $\neg P \vee Q \vee R$

(c) **Sum of products:** $(P \wedge Q) \vee (\neg P \wedge Q)$

(d) **Product of sums:** $(P \vee Q) \wedge (\neg P \vee Q)$

(e) **Minterm:** Product consisting of all variables

(f) **Maxterm:** Sum consisting of all variables

15. Principal Conjunctive Normal Form (PCNF) of a formula P is an equivalent formula consisting of maxterms only. That is,

P = Product of maxterms

$= (. \vee . \vee .....) \wedge (.... \vee ....) \wedge .... \wedge (.... \vee ....)$

For instance,

$P = P \wedge f = P \wedge (Q \vee \neg Q)$

$= (P \vee Q) \wedge (P \vee \neg Q)$

= products.of sums

16. Principal Disjunctive Normal Form: (PDNF) of a formula P is an equivalent formula consisting of minterms only.

For example,

$P = P \wedge t = P \wedge (Q \vee \neg Q)$

$= (P \wedge Q) \vee (P \wedge \neg Q)$

= sum of products

17. Infix notation: (formula 1) (connective) (formula 2) is called an infix notation. Example: $P \wedge Q$

18. Prefix notation: (connective) (formula 1) (formula 2) is called a prefix notation. Example: $\wedge PQ$

19. Postfix notation: (formula 1) (formula 2) (connective) is known as postfix notation. Example, $PQ\wedge$.

20. Premises, formal proof, conclusion: An **argument** is an assertion that a given set of propositions $P_1, P_2, ..., P_n$, known as **Premises,** gives rise to another proposition Q, called the **conclusion**. The process of arriving at the conclusion from the premises by employing the rules of reasoning is known as **formal proof. A valid argument** is one in which from true premises, a true conclusion follows. An argument which is not valid is called a **fallacy.**

To arrive at a conclusion from a given set of premises taken to be true, the following two rules are used:-

Rule P: Any premise can be introduced at any stage.

Rule T: Any formula S can be introduced if S is obtained at a previous stage.

21. Inconsistency: The set of formulae $H_1, H_2, ..., H_n$ is said to be **inconsistent,** if $H_1 \wedge H_2 \wedge ... \wedge H_n$ is false.

22. Predicate: A proposition described in subject-predicate form is symbolized by a **predicate variable** or **propositional form** P(x) where P denotes the **predicate** and x, the subject. For example, if P(x) stands for "x is an integer" then "is an integer" is the predicate.

23 Universal quantifier and Existential quantifier: The symbols $\forall$ and $\exists$ are called **quantifiers** and they mean **"for all"** and **"there exists"** respectively. They are respectively known as the **universal quantifier** and the **existential quantifier**. Thus $\forall$ x P (x) is to be read as 'for all x (in its domain) P(x)' and, $\exists$ x P(x) means 'there exists x such that P(x)' or 'for some x, P(x)'. In x P(x), there are two x's; the x in P(x) is known as a **free variable** and the other $\forall$x is called a **bound variable** (or an **apparent variable**). The scope of a quantifier occurring in a formula is the formula to which the quantifier applies. For instance, in $\forall$ x P(x), the scope of the quantifiers $\forall$ x is P(x).

24. Universe of discourse or **domain** is the set of all values taken by a variable.

25. Rules of predicate calculus

Rule US (Universal Specification) $\forall x\, A(x) \Rightarrow A(y)$

Rule ES (Existential Specification) $\exists x\, A(x) \Rightarrow A(y)$, y is new.

Rule EG (Existential Generalization) $A(x) \Rightarrow \exists y\, A(y)$

Rule UG (Universal Generalization) $A(x) \Rightarrow \forall y\, A(y)$

If x is not free in the given premises and if x is free resulted from ES, then no variables introduced by that use of ES appear free in A(x).

26. Substitution instance (or rule) In formula $P \rightarrow Q$ substitute $(A \rightleftarrows B)$ for P and $(\neg R \vee S)$ for Q to get $(A \rightleftarrows B) \longrightarrow (\neg R \vee S)$.

27. Consistency Formulae $H_1, H_2, \ldots, H_n$ are said to be **consistent**, if they are not inconsistent.

Note

1: In propositional calculus the method of substitution and the truth table method are used to prove theorems. In predicate calculus the method of truth tables is not advantageous. The propositional calculus is complete in two senses namely (i) every tautology can be proved (ii) addition of an improvable proposition would read to contradiction (in the logical sense). But the predicate calculus is not complete in the logical sense; however, it is complete in the sense that every tautology is provable.

If a formula holds for some assignment of natural numbers to its individual variables and truth values for the predicate variables, then the formula is said to be **satisfiable** in the domain of natural numbers. For every formula A of the first order predicate calculus either $\neg A$ is provable or A is satisfiable in the domain of natural numbers. From the above result we have the following results:

(a) If a formula is universally valid, then it is provable.

(b) If a formula is valid in the domain of natural numbers, then it is valid in every domain.

(c) If a formula is satisfiable in some non-empty domain then it is satisfiable in the set of natural numbers.

2: Recall that a proposition is a statement involving only **definite** or **constant notions**. Thus 'The sky is blue', 'blood is red', 'man is mortal' are propositions whereas, all x's are y's, is not a proposition for it contains **variables** or **undetermined constituents**. When specific values are assigned to the variables in statements then they become propositions and are called **propositional functions**. Thus all x's are y's becomes the proposition 'all men are mortal' by assigning values to x and y. Functions are said to be **formally equivalent** if for the same values of their variables they are true or false.

Problems with Solutions (on connectives)

Problem: Given that P and Q are true; R and S are false, find the truth value of $(\neg(P \wedge Q) \vee \neg R) \vee (((\neg P \wedge Q) \vee \neg R) \wedge S)$

Solution: Using the truth values of P, Q, R, S, the given expression becomes

$$(\neg(T \wedge T) \vee \neg F) \vee (((\neg T \wedge T) \vee \neg F) \wedge F)$$
$$\equiv (\neg T \vee T) \vee (((F \wedge T) \vee T) \wedge F) \quad (\because T \wedge T = T, \neg T = F, \neg F = T)$$
$$\equiv (F \vee T) \vee ((F \vee T) \wedge F) \quad (\because F \wedge T \equiv F)$$
$$\equiv T \vee (T \wedge F) \quad (\because F \vee T \equiv T)$$
$$\equiv T \vee F$$
$$\equiv \mathbf{T}$$

$\therefore$ the truth value of the given expression is **T**.

Problem : Given that P and Q have truth values T; R and S have truth values F, find the truth value of $(P \vee (Q \rightarrow (R \wedge \neg P))) \rightleftarrows (Q \vee S)$

Solution: Applying the truth values the given expression becomes

$$(T \vee (T \rightarrow (F \wedge \neg T))) \rightleftarrows (T \vee \neg F)$$
$$\equiv (T \vee (T \rightarrow (F \wedge F))) \rightleftarrows (T \vee T)$$
$$\equiv (T \vee (T \rightarrow F)) \rightleftarrows T \quad (\because F \wedge F \equiv F, T \vee T \equiv T)$$
$$\equiv (T \vee F) \rightleftarrows T \quad (\because T \rightarrow F \equiv F)$$
$$\equiv T \rightleftarrows T$$
$$\equiv T$$

Thus the truth value of the given expression is **T**.

Problem : Show that the truth value of the formula $(P \rightarrow Q) \rightleftarrows (\neg P \vee Q)$ is independent of its components.

Solution: Method I

Constructing the truth table 1 for the formula,

Table 1

P	Q	$(P \to Q)$	$\rceil P$	$(\rceil P \vee Q)$	$(P \to Q) \rightleftarrows (\rceil P \vee Q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

From the last column, we find that $(P \to Q) \rightleftarrows (\rceil P \vee Q)$ is a **tautology** and so it is independent of its components.

Method II

$(P \to Q) \rightleftarrows (\rceil P \vee Q) \equiv (P \to Q) \rightleftarrows (P \to Q)$ ($\because$ $P \to Q$ and $\rceil P \vee Q$ have the same truth values)

$\equiv T$ as $P \rightleftarrows P$ **is always true.**

Problem : Find an equivalent formula for $\veebar$ consisting of $\wedge$, $\vee$, $\rceil$.

Solution: The connective $\veebar$ (exclusive OR) having an equivalent formulation can be read from the following truth table 2.

Table 2

P (1)	Q (2)	$P \veebar Q$ (3)	$P \wedge \rceil Q$ (4)	$\rceil P \wedge Q$ (5)	$(P \wedge \rceil Q) \vee (\rceil P \wedge Q)$ (6)
T	T	F	F	F	F
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	F	F	F

By columns (3) and (6), we observe that

$$P \veebar Q \equiv (P \wedge \rceil Q) \vee (\rceil P \wedge Q)$$

Problem : Show that $(P \wedge (P \to Q)) \to Q$ is a tautology.

Solution: We verify the statement by constructing the following truth table 3.

Table 3

P	Q	$P \to Q$	$P \wedge (P \to Q)$	Q	$(P \wedge (P \to Q)) \to Q$
T	T	T	T	T	T
T	F	F	F	F	T
F	T	T	F	T	T
F	F	T	F	F	T

The last column of the table shows that $(P \wedge (P \to Q)) \to Q$ is a tautology.

Problem : Prove the law of syllogism "If P implies Q and Q implies R, then P implies R".

Table 4

P (1)	Q (2)	R (3)	$(P \to Q)$ (4)	$(Q \to R)$ (5)	$(P \to R)$ (6)	$(P \to Q) \wedge (Q \to R)$ $(4) \wedge (5)$ (7)	$(7) \to (6)$ (8)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Solution: Since there are three variables P, Q ,R the truth table will contain $2 \times 2 \times 2 = 8$ lines. **(See Table 4)**

The last column shows that the given proposition is a tautology.

Problem: Show that P implies $Q \to P$.

Solution: Assume that P does not imply $Q \to P$. That is, $P \Rightarrow (Q \to P)$ is false. So P is true and $Q \to P$ is false. Now $Q \to P$ is false only when Q is true and P is false. Thus we find that P is true and P is false at the same time, a contradiction. $\therefore P \Rightarrow (Q \to P)$.

Problem: Prove $(Q \to (P \wedge \neg P)) \to$
$(R \to (P \wedge \neg P)) \Rightarrow (R \to Q)$

Solution: Assume that the given statement is false. Then $(Q \to (P \wedge \neg P)) \to (R \to (P \wedge \neg P))$ is true and $R \to Q$ is false. ... (1)

Since $R \to Q$ is false, R is true and Q is false. Now prepare the following truth table 5 observing that $P \wedge \neg P$ is always false.

Table 5

P	Q	R	$P \wedge \neg P$	$Q \to P \wedge \neg P$ (1)	$R \to P \wedge \neg P$ (2)	(1)→ (2)
T	F	T	F	T	F	F
F	F	T	F	T	F	F

The last column of the table shows that $(Q \to (P \wedge \neg P)) \to (R \to (P \wedge \neg P))$ is always false. This contradicts (1). Hence our assumption is wrong. So the given statement must be true.

Problem: Show that $\neg(P \rightleftarrows Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Solution

The result follows from the **Table 6**.

Problem: Show that $\{\wedge, \vee\}$ is not functionally complete.

Solution: $P \vee \neg P$, which is always true, cannot be written in terms of $\vee$ and $\wedge$ only. Hence $\{\wedge, \vee\}$ is not functionally complete.

Problem : Show that $\{\uparrow\}$ is functionally complete.

Solution: We know

$$P \uparrow P \equiv \neg(P \wedge P) \equiv \neg P$$

$$(P \uparrow Q) \uparrow (P \uparrow Q) \equiv \neg(P \uparrow Q) \wedge (P \uparrow Q)$$
$$\equiv \neg(\neg(P \wedge Q) \wedge \neg(P \wedge Q))$$
$$\equiv \neg(\neg(P \wedge Q))$$
$$\equiv P \wedge Q \text{ and}$$

$$(P \uparrow P) \uparrow (Q \uparrow Q) \equiv \neg((P \uparrow P) \wedge (Q \uparrow Q))$$
$$\equiv \neg(\neg(P \wedge P) \wedge \neg(Q \wedge Q))$$
$$\equiv \neg(\neg P \wedge \neg Q)$$
$$\equiv P \vee Q$$

Hence any formula consisting of $\wedge$, $\vee$ and $\neg$ can be written as a formula consisting of $\uparrow$ only. Thus $\{\uparrow\}$ is functionally complete.

Problem : Express $P \to (\neg P \to Q)$ in terms of $\uparrow$.

Solution

$$P \to (\neg P \to Q) \equiv \neg P \vee (\neg P \to Q)$$
$$(\because P \to Q \equiv \neg P \vee Q)$$
$$\equiv \neg P \vee (\neg\neg P \vee Q)$$
$$\equiv \neg P \vee (P \vee Q)$$
$$\equiv \neg P \vee P \vee Q$$
$$\equiv \neg P \vee P$$
$$(\because \neg P \vee P \text{ is always true, } \neg P \vee P \vee Q \equiv t)$$
$$\equiv \neg(P \wedge \neg P)$$
$$\equiv P \uparrow \neg P.$$

Problem : Express $P \uparrow Q$ in terms of $\downarrow$.

Solution

$$P \uparrow Q \equiv \neg(P \wedge Q) \equiv \neg((P \wedge Q) \vee (P \wedge Q))$$

Table 6

P	Q	$P \rightleftarrows Q$	$\neg(P \rightleftarrows Q)$ (1)	$\neg Q$	$P \wedge \neg Q$ (2)	$\neg P$	$\neg P \wedge Q$ (3)	(2) ∨ (3) = (4)	(1) ⇔ (4)
T	T	T	F	F	F	F	F	F	T
T	F	F	T	T	T	F	F	T	T
F	T	F	T	F	F	T	T	T	T
F	F	T	F	T	F	T	F	F	T

$\equiv (P \wedge Q) \downarrow (P \wedge Q)$ ... (1)

Now, $P \wedge Q \equiv (P \vee P) \wedge (Q \vee Q)$

$\equiv \neg(P \downarrow P) \wedge \neg(Q \downarrow Q)$

$\equiv \neg((P \downarrow P) \vee (Q \downarrow Q))$

$\equiv (P \downarrow P) \downarrow (Q \downarrow Q)$

Using this in (1),

$P \uparrow Q \equiv ((P \downarrow P) \downarrow (Q \downarrow Q)) \downarrow ((P \downarrow P) \downarrow (Q \downarrow Q))$.

Problem : Show that the following statements are inconsistent:

$P \to Q$, $P \to R$, $Q \to \neg R$ and P.

Solution

By the law of syllogism **(See problem worked out in our earlier page).**

$P \to Q$ and $Q \to \neg R$ imply $P \to \neg R$. Also given $P \to R$ and P. Now

Table 7

P (1)	R	$\neg R$	$P \to R$ (2)	$P \to \neg R$ (3)
T	T	F	T	F
T	F	T	F	T
F	T	F	T	T
F	F	T	T	T

A scrutiny of columns (1), (2) and (3) of the above table 7 shows that the given statements are **inconsistent**.

Problem : Prove that C follows from H_1 and H_2 where $H_1 : P \to (Q \to R)$, $H_2 : P \wedge Q$, $C : R$

Solution: We have to show $(H_1 \wedge H_2) \Rightarrow C$. That is, if $H_1 \wedge H_2$ is true then C is true.

Suppose that $H_1 \wedge H_2$ is true. i.e., both H_1 and H_2 are true.

Then $P \to (Q \to R)$ and $P \wedge Q$ are true.

i.e., $P \to (Q \to R)$, P and Q are true.

Since $P \to (Q \to R)$ is true and P is true, $Q \to R$ must be true. But Q is true $\therefore$ R is true. i.e., C is true.

Problem : Without constructing a truth table show that $A \wedge E$ is not a valid consequence of

$H_1: A \rightleftarrows B$, $H_2: B \rightleftarrows (C \wedge D)$, $H_3: C \rightleftarrows (A \vee E)$, $H_4: A \vee E$.

Solution: Note that to prove Q is not a valid consequence of P it is enough to prove that P is true while Q is false.

For the combinations A, B, D false and C, E true we first check that H_1, H_2, H_3, H_4 are true.

Since A, B are both false $H_1 : A \rightleftarrows B$ is true.

Since C is true and D is false, $C \wedge D$ is false. Also B is false.

$\therefore H_2: B \rightleftarrows C \wedge D$ is true.

Again, A is false, E is true imply $A \vee E$ is true. But C is true.

$\therefore H_3 : C \rightleftarrows (A \vee E)$ is also true.

Lastly $H_4 : A \vee E$ is true.

Now A is false while E is true. So $A \wedge E$ is false.

Thus $H_1 \wedge H_2 \wedge H_3 \wedge H_4$ is true and $A \wedge E$ is false.

So $A \wedge E$ is not a valid conclusion of H_1, H_2, H_3, H_4.

Problem : Find the simplest form of $((P \to Q) \rightleftarrows (\neg Q \to \neg P)) \wedge R$

Solution: First note that $P \to Q$ is equivalent to $\neg Q \to \neg P$.

Table 8

P	Q	$P \to Q$ (1)	$\neg Q$	$\neg P$	$\neg Q \to \neg P$ (2)	$(1) \rightleftarrows (2)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Thus $(P \to Q) \rightleftarrows (\neg Q \to \neg P)$ is always true.

So the simplest form of $((P \to Q) \rightleftarrows \neg Q \to \neg P) \wedge R$ is R (by identity law).

Problem : Find the conclusion from the premises $(A \to B) \wedge (A \to C)$, $(B \wedge C)$, $(D \vee A)$.

Solution: By rule P, consider $(A \to B) \wedge (A \to C)$.

Then $A \to B \wedge C$ by rule T.

Now consider $\neg(B \wedge C)$ by rule P.

Since $A \to B \wedge C$. $\neg(B \wedge C) \to \neg A$. Thus we get $\neg A$.

Taking $D \vee A$ with $\neg A$, we conclude D.

Problem : Obtain the principal conjunctive normal form and the principal disjunctive normal form of $P \vee (\neg P \to (Q \vee (\neg Q \to R)))$.

Solution: We know that $P \to Q$ is equivalent to $\neg P \vee Q$.

$\therefore$ So $\neg Q \to R \equiv Q \vee R$.

Hence $Q \vee (\neg Q \to R) \equiv Q \vee (Q \vee R)$

$\equiv Q \vee R$

Thus $A \equiv P \vee (\rceil P \rightarrow (Q \vee (\rceil Q \rightarrow R)))$

$\equiv P \vee (\rceil P \rightarrow Q \vee R)$

$\equiv P \vee (P \vee Q \vee R)$

$\equiv P \vee Q \vee R$ is the required principal conjunctive normal form.

Let $\overline{P}$ denote $\rceil P$ and $\overline{A}$ stand for the product of the remaining $\rceil$ maxterms.

Then

$$\overline{A} \equiv (P \vee Q \vee \overline{R}) \wedge (P \vee \overline{Q} \vee R) \wedge (\overline{P} \vee Q \vee R) \wedge (P \vee \overline{Q} \vee \overline{R})$$

$$\wedge (\overline{P} \vee Q \vee \overline{R}) \wedge (\overline{P} \vee \overline{Q} \vee R) \wedge (\overline{P} \vee \overline{Q} \vee \overline{R})$$

So,

$$A = \overline{\overline{A}} \equiv (\overline{P} \wedge \overline{Q} \wedge R) \vee (\overline{P} \wedge Q \wedge \overline{R}) \vee (P \wedge \overline{Q} \wedge \overline{R}) \vee (\overline{P} \wedge Q \wedge R)$$

$$\vee (P \wedge \overline{Q} \wedge R) \vee (P \wedge Q \wedge \overline{R}) \vee (P \wedge Q \wedge R)$$

is the required principal disjunctive normal form.

Problem : Write $P \wedge \rceil R \rightarrow Q \rightleftarrows P \wedge Q$ in (i) prefix form (ii) postfix form.

Solution: (i) The precedence is $\rceil, \wedge, \vee, \rightarrow, \rightleftarrows$ where $\rceil$ has the highest precedence.

Now $A \equiv P \wedge \rceil R \rightarrow Q \rightleftarrows P \wedge Q$

$\equiv ((P \wedge \rceil R) \rightarrow Q) \rightleftarrows (P \wedge Q)$

$\equiv \rightleftarrows \rightarrow \wedge P \rceil RQ \wedge PQ$

(see **Figure** below)

$A \equiv ((P \wedge \rceil R) \longrightarrow Q) \rightleftarrows (P \wedge Q)$

Prefix notation

(ii) Consider **Figure** below.

$A \equiv ((P \wedge (\rceil R)) \longrightarrow Q) \rightleftarrows (P \wedge Q)$

Postfix notation

$\therefore A \equiv PR\rceil \wedge Q \longrightarrow PQ\wedge \rightleftarrows$

Problem: Convert $P \rceil P \rightarrow P \rightarrow P \rightarrow$ into infix notation.

Solution: The given expression is in postfix notation. Moving from left to right,

$A \equiv P \rceil P \rightarrow P \rightarrow P \rightarrow$

$\equiv (\rceil P) P \rightarrow P \rightarrow P \rightarrow$

$\equiv (\rceil P \rightarrow P) P \rightarrow P \rightarrow$

$\equiv ((\rceil P \rightarrow P) \rightarrow P) P \rightarrow$

$\equiv ((\rceil P \rightarrow P) \rightarrow P) \rightarrow P$

Problem: For the conditional $P \rightarrow Q$, the propositions $Q \rightarrow P$, $\rceil P \rightarrow \rceil Q$ and $\rceil Q \rightarrow \rceil P$ are called the **converse, inverse** and **contra positive** respectively. Which of these are logically equivalent to $P \rightarrow Q$?

Solution: Constructing the truth table 9 we find that the contra positive $\rceil Q \rightarrow \rceil P$ is logically equivalent to $P \rightarrow Q$.

Table 9

P	Q	$\rceil P$	$\rceil Q$	$P \rightarrow Q$	$Q \rightarrow P$	$\rceil P \rightarrow \rceil Q$	$\rceil Q \rightarrow \rceil P$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

We now list some implications and equivalences which can be easily established.

1. $P, Q \Rightarrow P \wedge Q$
2. $P \wedge Q \Rightarrow P$
3. $P \Rightarrow P \vee Q$
4. $\rceil P \Rightarrow P \rightarrow Q$
5. $Q \Rightarrow P \rightarrow Q$
6. $\rceil (P \rightarrow Q) \Rightarrow P$
7. $\rceil (P \rightarrow Q) \Rightarrow \rceil Q$
8. $P \wedge (P \rightarrow Q) \Rightarrow Q$
9. $\rceil Q \wedge (P \rightarrow Q) \Rightarrow \rceil P$
10. $\rceil P \wedge (P \vee Q) \Rightarrow Q$
11. $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \Rightarrow R$
12. $\rceil (P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \rceil Q$

OBJECTIVE QUESTIONS

1. One of the following is a declarative statement. Identify it.

(a) It is beautiful

(b) He says "it is correct"

(c) Two may not be an even integer

(d) I love you

2. $P \rightarrow (Q \rightarrow R)$ is equivalent to

(a) $(P \wedge Q) \rightarrow R$ **(b)** $(P \vee Q) \rightarrow R$

(c) $(P \vee Q) \rightarrow \rceil R$ **(d)** $(P \wedge Q) \rightarrow \rceil R$

3. $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)$ is equivalent to

(a) P **(b)** Q

(c) R **(d)** $\rceil P$

4. $\rceil (P \rightarrow Q)$ is equivalent to

(a) $P \wedge \rceil Q$ **(b)** $\rceil P \wedge Q$

(c) $\rceil P \vee \rceil Q$ **(d)** $P \vee Q$

5. Choose the functionally complete set.

(a) $\{\rceil, \wedge, \vee\}$ **(b)** $\{\downarrow, \wedge\}$

(c) $\{\uparrow\}$ **(d)** $\{\rightarrow, \wedge\}$

6. Pick the **correct** prefix formula.

(a) $\rightarrow \neg P \vee Q \rightleftarrows \neg R \neg S$

(b) $\rightarrow P \neg Q \vee \rightleftarrows \neg RS$

(c) $\rightarrow \rightarrow PQ \rightarrow \rightarrow QR \rightarrow PR$

(d) $\rightarrow \neg PV \rightleftarrows QSP$

7. Which one of the following statements is true when exactly two of P, Q or R are true? ($\overline{P}$ means $\neg P$)

(a) $(\overline{P}\wedge\overline{Q}\wedge R)\vee(P\wedge\overline{Q}\wedge\overline{R})\wedge(\overline{P}\wedge Q\wedge\overline{R})$

(b) $(\overline{P}\vee Q\vee R)\wedge(P\vee\overline{Q}\vee R)\wedge(P\vee Q\vee\overline{R})$

(c) $(\overline{P}\wedge Q\wedge R)\vee(P\wedge\overline{Q}\wedge R)\vee(P\wedge Q\wedge\overline{R})$

(d) $(\overline{P}\wedge\overline{Q}\wedge R)\vee(P\wedge\overline{Q}\wedge\overline{R})\wedge(\overline{P}\wedge Q\wedge\overline{R})$

8. Say true or false.
The two formulae $((P\wedge Q) \vee (P\wedge R)) \rightarrow S$ and $(\overline{P} \vee (\overline{Q} \wedge \overline{R}) \vee S$ are equivalent.

9. The negation of the statement $\exists x \ \forall y, P(x, y)$ is

(a) $\forall x \exists y, P(x, y)$

(b) $\forall x \exists y, \neg P(x, y)$

(c) $\exists x \exists y, \neg P(x, y)$

(d) $\forall y \forall x, \exists z \neg P(x, y, z)$

10. Which one of the following statements is true?

1. A valid argument is one in which from true premises a true conclusion follows.
2. The truth-table method provides us with a method of deciding whether a formula is a tautology or not.
3. The propositional logic is sufficient to meet the demands of mathematics.

(a) 1, 2 (b) 2, 3
(c) 1, 3 (d) All the three

11. Which of the following statements is false?

1. First order logic (predicate calculus) is one in which the predicate variables themselves are not quantified; that is, forms like $\forall P(x)$ do not occur in this logic.
2. Principal conjunctive normal form of a formula is an equivalent formula consisting of minterms only.

(a) 1 only (b) 2 only
(c) both 1 and 2 (d) neither 1 nor 2

12. The compound proposition (last column) corresponding to the truth table

p	q	$\neg q$	$p \vee \neg q$	?
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

(a) $p \rightarrow q$ (b) $p \rightarrow \neg q$
(c) $p \vee \neg q \rightarrow q$ (d) $q \rightarrow p$

13. Which of the following statements is/are true?

1. The propositional calculus is a **decidable theory.**
2. The propositional calculus is sufficient to meet the demands of mathematics.
3. The propositional calculus is **complete** in the sense that every tautology can be proved.

(a) 1 and 2 (b) 2 and 3
(c) 1 and 3 (d) All the three

14. Identify the tautology which is known as **the Law of the Excluded Middle**
1. $P \vee \neg P$ 2. $\neg [P \vee (\neg P)]$

(a) 1 (b) 2
(c) neither 1 nor 2 (d) both 1 and 2

15. Identify the tautology which is known as **the Law of contradiction.**
1. $\neg [P \vee (\neg P)]$
2. $P \vee \neg P$

(a) 1 (b) 2
(c) both 1 and 2 (d) neither 1 nor 2

16. If the proposition $\neg P \Rightarrow q$ is true then the truth value of the proposition $\neg P \vee (p \Rightarrow q)$ is

(a) true (b) multiple-valued
(c) false (d) cannot be determined

17. Let a, b, c, d be propositions. Assume that the equivalences $a \leftrightarrow (b \vee \neg b)$ and $b \leftrightarrow c$ hold. Then the truth value of the formula $(a \wedge b) \longrightarrow ((a \wedge c) \vee d)$ is always

(a) true
(b) false
(c) same as the truth value of b
(d) same as the truth value of d

18. An equivalent formula for
$(p \vee q) \wedge r \Rightarrow (p \vee r)$
containing the connectives $\wedge$ and $\neg$ only is

(a) $p \wedge \vee p$ (b) $\vee (p \wedge \vee p)$
(c) $\vee (p \wedge r)$ (d) none of these

19. Domain: The set of all human beings
$\phi(x, y) \equiv$ 'y is the father of x'
Which of the following propositions is true?
1. $(\forall x)(\exists y)\ \phi(x, y)$
2. $(\exists y)(\forall x)\ \phi(x, y)$
(a) 1 only **(b)** 2 only
(c) both 1 and 2 **(d)** neither 1 nor 2

20. Which of the following statements is true?
1. Quantification eliminates variables from propositional functions.
2. The type of proof in which a preposition p is shown to be false by showing that p leads to on absurdity is known as **reductio and absurdum**.
(a) 1 **(b)** 2
(c) both 1 and 2 **(d)** neither 1 nor 2

KEY

1. (b)	2. (a)	3. (c)	4. (a)
5. (c)	6. (c)	7. (c)	8. True
9. (b)	10. (a)	11. (b)	12. (c)
13. (c)	14. (a)	15. (a)	16. (b)
17. (a)	18. (b)	19. (a)	20. (c)

EXERCISE

1. Let p and q be the propositions.
p : It is below freezing
q : It is snowing
Write the following propositions using p, q and logical connectives.
(a) It is below freezing and snowing
(b) It is below freezing but not snowing
(c) It is not below freezing and not snowing
(d) It is either snowing or below freezing or both
(e) If it is below freezing, it is not snowing
(f) It is either below freezing or it is snowing but it is not snowing if it is below freezing
(g) That it is below freezing is necessary and sufficient for it to be snowing

2. Construct a truth table for each of the following compound propositions:-
(a) $p \wedge \neg p$
(b) $p \vee \neg p$
(c) $(p \vee \neg q) \rightarrow q$
(d) $(p \vee q) \rightarrow (p \wedge q)$
(e) $(p \rightarrow q) \rightleftarrows (\neg q \rightarrow \neg p)$
(f) $(p \rightarrow q) \rightarrow (q \rightarrow p)$
(g) $p \rightarrow (\neg q \vee r)$
(h) $\neg p \rightarrow (q \rightarrow r)$
(i) $(p \rightarrow q) \vee (\neg p \rightarrow r)$
(j) $(p \rightarrow q) \wedge (\neg p \rightarrow r)$
(k) $(p \rightarrow q) \vee (\neg q \rightleftarrows r)$
(l) $(\neg p \rightleftarrows \neg q) \rightleftarrows (q \rightleftarrows r)$

3. Show that the propositions $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. Find if $(p \wedge q) \rightarrow (p \vee q)$ is a tautology or a contradiction.

4. Show that **(i)** $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent.
(ii) $\neg(p \rightleftarrows q)$ and $\neg p \rightleftarrows q$ are logically equivalent.

5. Obtain an equivalent proposition to $p \rightarrow q$ using the operator $\downarrow$ only.

6. Will $\neg$, $\wedge$ and $\vee$ form a functionally complete collection of logical operators?

7. The truth value of negation of a proposition in fuzzy logic is 1 minus the truth value of the proposition. What are the truth values of the statements?
"Ram is not happy" and "Krishna is not happy"
(Given that the truth value for the statement "Ram is happy" is 0.8 and the truth value for the statement "Krishna is happy" is 0.4.)

8. Find the bitwise OR, bitwise AND and bitwise XOR of each of the following pairs of bit strings
(a) 1011110, 0100001
(b) 111 10000, 10101010
(c) 00011 10001, 1001001000
(d) 11111 11111, 00000 00000

PROBLEMS WITH SOLUTIONS

1. Given that P, Q, R, S have truth values T, T, F, F respectively, find the truth value of
$(\neg (P \wedge Q) \vee \rceil R) \vee ((Q \rightleftarrows \neg P) \rightarrow (R \vee \neg S))$.

Solution

Substituting the truth values of P, Q, R, S in the given statement, we get

$(\neg (T \wedge T) \vee T) \vee ((T \rightleftarrows F) \rightarrow (F \vee T))$

$\equiv (F \vee T) \vee (F \rightarrow T)$

$\equiv T \vee T$

$\equiv T$

$\therefore$ the given statement has the truth value T.

2. Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$ is equivalent to R.

Solution

$\neg P \wedge \neg Q \equiv \neg (P \vee Q)$ and $(Q \wedge R) \vee (P \wedge R)$

$\equiv (P \vee Q) \wedge R$

$\therefore$ the given statement

$\equiv (\neg (P \vee Q) \wedge R) \vee ((P \vee Q) \wedge R)$

$\equiv (\neg (P \vee Q) \vee (P \vee Q)) \wedge R$

$\neg (P \vee Q) \vee (P \vee Q)$ is a tautology.

$\therefore$ the statement is equivalent to R.

3. Prove the following.

(a) If $H_1, H_2, \ldots\ldots, H_n$ and P imply Q, then $H_1, H_2, \ldots\ldots, H_n$ imply $P \rightarrow Q$

(b) $P \rightarrow (Q \rightarrow R) \Rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$

(c) $(P \rightarrow Q) \rightarrow Q \Rightarrow P \vee Q$

(d) $\neg (P \rightleftarrows Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Solution

(a) Given $(H_1 \wedge H_2 \wedge \ldots\ldots \wedge H_n \wedge P) \Rightarrow Q$ ... (1)

To prove $(H_1 \wedge H_2 \wedge \ldots\ldots \wedge H_n) \Rightarrow P \rightarrow Q$

Assume that $H_1 \wedge H_2 \wedge \ldots\ldots \wedge H_n$ is true and $P \rightarrow Q$ is false.

i.e., $H_1, H_2, \ldots\ldots, H_n$ are true; P is true, R is false.

$\therefore H_1 \wedge H_2 \wedge \ldots\ldots \wedge H_n \wedge P$ is true, Q is false.

$\therefore H_1 \wedge H_2 \wedge \ldots\ldots \wedge H_n \wedge P$ Q which contradicts (1) and so the result follows.

(b) Construct the truth table.**(Table 10)**

From the last column of the table, the result follows.

Table 11

(c)

P	Q	P → Q	(P → Q) → Q (1)	P ∨ Q (2)	(1) ⇒ (2)
T	T	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	F	F	T

The last column proves the result.

(d) (Table 12)

Clearly from the table 12 (1) ⇔(4).

Table 10

P (1)	Q	R	Q → R (2)	(1) → (2) (3)	P → Q (4)	P → R (5)	(4) → (5) (6)	(3) → (6)
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Table 12

P	Q	$P \rightleftarrows Q$	$\neg(P \rightleftarrows Q)$ (1)	$\neg Q$	$P \wedge \neg Q$ (2)	$\neg P$	$\neg P \wedge Q$(3)	(2) $\vee$ (3)= (4)
T	T	T	F	F	F	F	F	F
T	F	F	T	T	T	F	F	T
F	T	F	T	F	F	T	T	T
F	F	T	F	T	F	T	F	F

4. Find an equivalent formula for

(i) $\neg(P \to (Q \wedge R))$ containing $\vee$ and $\neg$ only

(ii) $(P \rightleftarrows Q) \wedge (\neg P \to Q)$ containing $\neg$ and $\vee$ only

(iii) $(P \to Q) \vee R$ containing $\uparrow$ only

Solution

(i) We know that $A \wedge B = \neg(\neg A \vee \neg B)$ ... (1)

and $A \to B \equiv \neg A \vee B$

$\therefore \neg(P \to Q \wedge R) \equiv \neg\ (\neg P \vee (Q \wedge R))$
$\equiv \neg(\neg P \vee (\neg(\neg Q \vee \neg R)))$ (by (1))

(ii) We know that $P \rightleftarrows Q \equiv (P \to Q) \wedge (Q \to P)$ and

$\neg P \to Q \equiv P \vee Q$

$\therefore (P \rightleftarrows Q) \wedge (\neg P \to Q) \equiv (P \to Q) \wedge (Q \to P) \wedge (P \vee Q)$
$\equiv (\neg P \vee Q) \wedge (\neg Q \vee P) \wedge (P \vee Q)$

$\equiv A \wedge B \wedge C$

where $A = \neg P \vee Q$, $B = \neg Q \vee P$ and $C = P \vee Q$

Note that $A \wedge B \wedge C = \neg(\neg A \vee \neg B \vee \neg C)$. Thus the equivalent formula contains only $\neg$ and $\vee$.

(iii) We know that $P \uparrow P = \neg(P \wedge P) = \neg P$ ... (1)

Also, $P \wedge Q \equiv (P \uparrow Q) \uparrow (P \uparrow Q)$ ... (2)

and $P \vee Q \equiv (P \uparrow P) \uparrow (Q \uparrow Q)$ ... (3)

$\therefore (P \to Q) \vee R \equiv (\neg P \vee Q) \vee R$

$\equiv ((\neg P \vee Q) \uparrow (\neg P \vee Q)) \uparrow (R \uparrow R)$ (by (3))

$\equiv ((\neg P \uparrow \neg P) \uparrow (Q \uparrow Q)) \uparrow ((\neg P \uparrow \neg P) \uparrow (Q \uparrow Q)) \uparrow (R \uparrow R)$ (by (3))

$\equiv (P \uparrow (Q \uparrow Q)) \uparrow (P \uparrow (Q \uparrow Q)) \uparrow (R \uparrow R)$ (by (1)).

5. Show that $\{\neg, \to\}$ is functionally complete.

Solution

Recall that a set of connectives is said to be functionally complete if any formula can be written as an equivalent formula containing these connectives only. Obviously $\{\neg, \vee, \wedge\}$ is functionally complete.

Since $P \wedge Q \equiv \neg(\neg P \vee \neg Q)$, $\{\neg, \vee\}$ is functionally complete.

Since $P \vee Q \equiv \neg P \to Q$, wherever $\vee$ occurs we can substitute for $\vee$ the connectives $\neg$ and $\to$.

$\therefore$ $\{\neg, \vee\} = \{\neg, \to\}$, in turn, implies that $\{\neg, \to\}$ is functionally complete.

6. Prove that $((P \wedge Q \wedge R) \to S) \wedge (R \to (P \vee Q \vee S)) \Leftrightarrow (R \wedge (P \rightleftarrows Q)) \to S$

Solution

Left side of the biconditional

$\equiv ((P \wedge Q \wedge R) \to S) \wedge (R \to P \vee Q \vee S)$

$\equiv (\neg(P \wedge Q \wedge R) \vee S) \wedge (\neg R \vee (P \vee Q \vee S))$

$\equiv (\neg(P \wedge Q \wedge R) \vee S) \wedge ((\neg R \vee P \vee Q) \vee S)$

$\equiv (\neg(P \wedge Q \wedge R) \wedge (\neg R \vee P \vee Q)) \vee S$

$\equiv \neg((P \wedge Q \wedge R) \vee \neg(\neg R \vee P \vee Q)) \vee S$

$\equiv \neg((P \wedge Q \wedge R) \vee (R \wedge \neg(P \vee Q))) \vee S$

$\equiv ((P \wedge Q \wedge R) \vee (R \wedge \neg(P \vee Q))) \to S$

$\equiv ((R \wedge (P \wedge Q) \vee (R \wedge \neg(P \vee Q))) \to S$

$\equiv R \wedge ((P \wedge Q) \vee (\neg(P \vee Q))) \to S$

$\equiv (R \wedge (P \rightleftarrows Q)) \to S$

$(\because P \rightleftarrows Q \equiv (P \wedge Q) \vee (\neg P \vee \neg Q)$
$\equiv (P \wedge Q) \vee \neg(P \vee Q))$
$= \text{RHS}.$

7. Find the principal conjunctive normal form of $(P \wedge Q) \vee (\neg P \wedge Q)$.

Solution

Put $f = (P \wedge Q) \vee (\neg P \wedge Q)$ = sum of some min terms.

Then $\bar{f}$ = sum of remaining min terms

$$= (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$$

$$\therefore \quad f = \bar{\bar{f}} = \overline{(P \wedge \neg Q) \vee (\neg P \wedge \neg Q)}$$

$$= \overline{(P \wedge \neg Q)} \vee \overline{(\neg P \wedge \neg Q)}$$

$$= (\neg P \vee Q) \wedge (P \vee Q)$$

= product of sums.

8. Obtain the principal disjunctive normal form of $P \to (P \wedge (Q \to P))$

Solution

$P \to (P \wedge (Q \to P))$

$\equiv \neg P \vee (P \wedge (\neg Q \vee P))$

$\equiv \neg P \vee ((P \wedge \neg Q) \vee (P \wedge P))$

$\equiv \neg P \vee ((P \wedge \neg Q) \vee P)$

$\equiv P \vee \neg P \vee (P \wedge \neg Q)$

$\equiv t \vee (P \wedge \neg Q)$ ($\because P \vee \neg P \equiv t$, a tautology)

$\equiv t$

$\equiv (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

$\equiv$ sum of products (min terms).

9. Find the infix, prefix and suffix notation for $P \wedge \neg R \rightleftarrows P \vee Q$.

Solution

Put $f = P \wedge \neg R \rightleftarrows P \vee Q$

The procedure is $\neg, \wedge, \vee, \to, \rightleftarrows$

Infix $f \equiv (P \wedge \neg R) \rightleftarrows P \vee Q$

$\equiv (P \wedge \neg R) \rightleftarrows P \vee Q$

Prefix $(P \wedge \neg R) \rightleftarrows (P \vee Q)$

$\therefore f \equiv \rightleftarrows \wedge P \neg R \vee PQ$

Suffix $(P \wedge \neg R) \rightleftarrows (P \vee Q)$

$\therefore f \equiv PR \neg \wedge P Q \vee \rightleftarrows$

10. From $\neg P \vee Q$, $\neg (Q \wedge \neg R)$ and $\neg R$ conclude $\neg P$.

Solution

The premises are $\neg P \vee Q$, $\neg (Q \wedge \neg R)$ and $\neg R$. We use the rules of inference P and T.

	Rule of inference
1. $\neg R$	P
2. $\neg (Q \wedge \neg R)$	P
3. $\neg Q \vee R$	T ($\because \neg (P \wedge Q) \equiv \neg P \vee \neg Q$)
4. $\neg Q$	T ($\because \neg Q \vee R$ and $\neg R \Rightarrow \neg Q$)
5. $\neg P \vee Q$	P
6. $\neg P$	T ($\because \neg P \vee Q$ and $\neg Q \Rightarrow \neg P$)

$\therefore$ conclusion is $\neg P$.

11. Find whether R can be concluded from $P \vee Q$, $P \to R$ and $Q \to R$.

Solution

The premises are $P \vee Q$, $P \to R$ and $Q \to R$. Take $\neg R$ also as a premise. Use rules of inference.

	Rule of inference
1. $\neg R$	P
2. $Q \to R$	P
3. $\neg Q$	T ($\because Q \to R$ and $\neg R \Rightarrow \neg Q \vee R$ and $\neg R \Rightarrow \neg Q$)
4. $P \to R$	P
5. $\neg P$	T ($\because P \to R$ and $\neg R \Rightarrow \neg P \vee R$ and $\neg R \Rightarrow \neg P$)
6. $\neg P \wedge \neg Q$	T (by steps (5) and (3))
7. $\neg (P \vee Q)$	T
8. $P \vee Q$	P

We arrive at a contradiction from steps (7) and (8). So $\neg R$ cannot be a premise. i.e., R can be concluded from the given premises.

12. Show that $A \vee C$ is not a valid conclusion of $A \rightleftarrows (B \to C)$, $B \rightleftarrows (\neg A \vee \neg C)$, $C \rightleftarrows (A \vee \neg B)$ and B.

Solution

From the sixth row of the table 13 below, we find that all the four premises are true but the conclusion $A \vee C$ is false. $\therefore$ $A \vee C$ is not a valid conclusion.

Table 13

A	B	C	$\neg A$	$\neg B$	$\neg C$	$\neg A \vee \neg C$ (1)	$B \rightleftarrows$ (1)	$B \to C$ (2)	$A \rightleftarrows$ (2) (3)	$A \vee \neg B$ (4)	$C \rightleftarrows$ (4) (5)	$A \vee C$ (6)
T	T	T	F	F	F	F	F	T	T	T	T	T
T	T	F	F	F	T	T	T	F	F	T	F	T
T	F	T	F	T	F	F	T	T	T	T	T	T
T	F	F	F	T	T	T	F	T	T	T	F	T
F	T	T	T	F	F	T	T	T	F	F	F	T
F	$\boxed{T}$	F	T	F	T	T	$\boxed{T}$	F	$\boxed{T}$	F	$\boxed{T}$	$\boxed{F}$
F	F	T	T	T	F	T	F	T	F	T	T	T
F	F	F	T	T	T	T	F	T	F	T	F	F

13. Show that $A \to (B \to C)$, $D \to (B \wedge \neg C)$ and $A \wedge D$ are inconsistent.

Solution

Assume that the statements are consistent i.e., all are true.

$A \wedge D$ is true implies both A and D are true.

Since $A \to (B \to C)$ is true and A is true, $B \to C$ must be true.

Again $D \to (B \wedge \neg C)$ is true and D is true implies that $B \wedge \neg C$ is true; that both B and $\neg C$ are true or B is true and C is false. This means, $B \to C$ is false. Thus, we get a contradiction and so the given statements are inconsistent.

14. Show that $\neg (P \to Q) \to \neg (R \vee S)$, $((Q \to P) \vee \neg R)$, $R \Rightarrow (P \rightleftarrows Q)$.

Solution

We apply the rules of inference

	Rule of inference
1. R	P
2. $(Q \to P) \vee \neg R$	P
3. $Q \to P$	T (by (1) and (2))
4. $R \vee S$	T (since $R \Rightarrow R \vee S$)
5. $\neg (P \to Q) \to \neg (R \vee S)$	P
6. $P \to Q$	T (by (4) and (5))
7. $(P \to Q) \wedge (Q \to P)$	T (by (3) and (4))
8. $P \rightleftarrows Q$	T

15. Write a compound statement, which is true when none or one or two of the three variables P, Q, R are true.

Solution

It is given that a statement is true when none, or one or two of P, Q, R are true.

$\therefore$ statement $\equiv$

$$(\bar{P}\wedge\bar{Q}\wedge\bar{R})\vee(P\wedge\bar{Q}\wedge\bar{R})\vee(\bar{P}\wedge Q\wedge\bar{R})\vee$$
$$(\bar{P}\wedge\bar{Q}\wedge R)\vee(\bar{P}\wedge Q\wedge R)\vee(P\wedge\bar{Q}\wedge R)\vee$$
$$(P\wedge Q\wedge\bar{R})$$

2. PROBABILITY

2.1 MEASURES OF CENTRAL TENDENCY

These are the **Mean** (Arithmetic Mean A.M or the average), the **Median** (a posfitional average), the **Mode** (the value of the variable which occurs most frequently), the **Geometric Mean** (G.M. used extensively in finding the rate of population growth) and the **Harmonic Mean** (H.M.).

Mean, Median, Mode and Skewness

Let $x_1, x_2, \ldots\ldots, x_n$ be the observed values and let f_i be the frequency of (the x_i value of x_i repeated f_i times).

That is, if the data is 3, 1, 3, 3, 2, 1, 3, 2, 2 then $x_1=1, x_2=2$ and $x_3=3$, and $f_1=2, f_2=3$ and $f_3=4$.

Arithmetic Mean($\bar{x}$):

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots\ldots\ldots\ldots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and for frequency distribution

$$\bar{x} = \frac{1}{N}(f_1x_1 + f_2x_2 + \ldots\ldots\ldots\ldots + f_nx_n)$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_ix_i \quad \text{where } N = \sum_{i=1}^{n} f_i$$

Median: Median divides the collection of data into two equal parts. Hence this is called a **positional average**.

Problem : Find median of the following data.

Cost :	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
Items in a group :	4	5	3	6	3

Solution

Cost	No. of Items in the group	Cumulative frequency
10 – 20	4	4
20 – 30	5	9
30 – 40	3	12
40 – 50	6	18
50 – 60	3	21

Here N = 21. Hence N/2 = 10.5. The median class is 30 – 40. To find the exact median, we apply the formula

$$\text{Median} = 30 + \frac{10}{12}(10.5 - 9) = 30 + 1.25$$
$$= \mathbf{31.25}$$

Mode: Mode is the value which occurs most frequently.

$$\text{Median} = a + \frac{C(f_i - f_{i-1})}{2f_i - f_{i-1} - f_{i+1}}, \text{ where } (a - b)$$

is the modal class, f_i is the maximum frequency and C is the constant difference for each class.

Example: Find the mode for the following distribution.

Class interval	0 – 10	10 – 20	20 – 30	30 – 40
Frequency	5	9	8	12

Class interval	40 – 50	50 – 60	60 – 70	70 – 80
Frequency	28	20	12	11

Solution: Maximum frequency = 28

$\therefore$ modal class = 40 – 50

$$\text{Mode} = a + \frac{C(f_i - f_{i-1})}{2f_i - f_{i-1} - f_{i+1}}$$

$$= 40 + \frac{10(28 - 12)}{(2 \times 28) - 12 - 20}$$

$$= 40 + 6.666 = \mathbf{46.67}$$

Example : The expenditure of 100 families is given in the following frequency table.

Monthly Expenditure (in hundreds of rupees)	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
No. of families	14	?	27	?	15

If the mode for the distribution is 24, calculate the missing frequencies.

Solution: Let the missing frequencies for the classes 10 – 20 and 30 – 40 be f_1 and f_2 respectively.

Expenditure	No. of families	Cumulative frequency
0 – 10	14	14
10 – 20	f_1	$14 + f_1$
20 – 30	27	$41 + f_1$
20 – 30	f_2	$41 + f_1 + f_2$
40 – 50	15	$56 + f_1 + f_2$

$$N = 100 = 56 + f_1 + f_2$$

$$f_1 + f_2 = 100 - 56 = 44 \quad ...(1)$$

Mode is given to be 24, which lies in the class 20 – 30. So 20 – 30 is the modal class.

$$\text{Mode} = a + \frac{C(f_i - f_{i-1})}{2f_i - f_{i-1} - f_{i+1}}$$

$$24 = 20 + \frac{10(27 - f_1)}{(2 \times 27) - f_1 - f_2}$$

$$24 - 20 = \frac{270 - 10f_1}{54 - f_1 - f_2} = \frac{270 - 10f_1}{54 - 44}$$

$$(\because f_1 + f_2 = 44)$$

$\therefore f_1 = 23$

Substituting in (1), we get $f_2 = 21$.

Note: For a symmetrical distribution, Mean = Median = Mode and for a moderately asymmetrical distribution, Mode = 3 Median – 2 Mean or Mean – Mode = 3 (Mean – Median)

Geometric Mean (G)

$$G = \sqrt[n]{x_1 x_2 \dots x_n} = (x_1 x_2 \dots x_n)^{1/n}$$

and for a frequency distribution

$$G = \left(x_1^{f_1} x_2^{f_2} \dots x_n^{f_n}\right)^{\frac{1}{N}} \quad \text{where } N = \sum_{i=1}^{n} f_i$$

and for both the cases, taking logarithms, we get

$$\log G = \frac{1}{n} \sum_{i=1}^{n} \log(x_i)$$

and for a frequency distribution,

$$\log G = \frac{1}{N} \sum_{i=1}^{n} f_i \log(x_i)$$

Note 1: If n_1 and n_2 are the sizes and G_1 and G_2 are the geometric means of two series, then G, the geometric mean of the combined series is given by

$$\log G = \frac{n_1 \log G_1 + n_2 \log G_2}{n_1 + n_2} \text{(prove)}$$

Note 2: Geometric mean is used to find the rate of population growth and the rate of interest and it is also used in the construction of index numbers.

Harmonic Mean (H): Harmonic mean is the reciprocal of mean of reciprocals. That is,

$$H = 1 \div \left[\frac{1}{n} \sum_{i=1}^{n} (1/x_i)\right]$$

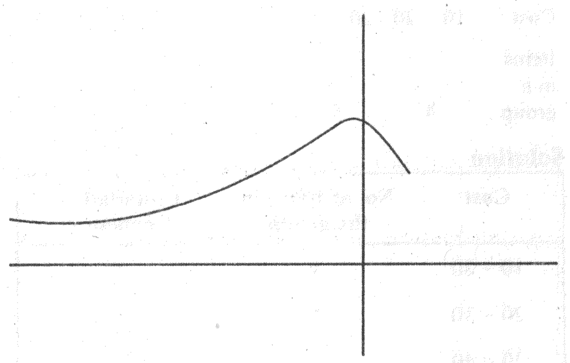
and for a frequency distribution,

$$H = 1 \div \left[\frac{1}{N} \sum_{i=1}^{n} (f_i/x_i)\right]$$

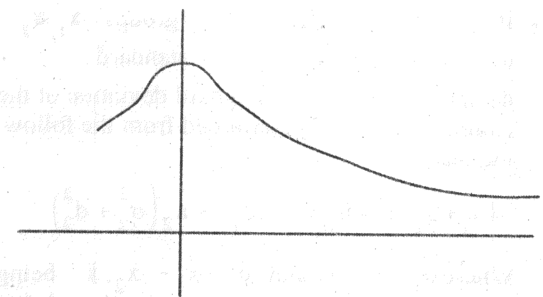
$$\text{where } N = \sum_{i=1}^{n} f_i$$

Skewness: Skewness is defined as **lack of symmetry**.

$$\text{Coefficient of skewness} = \frac{(\text{mean} - \text{mode})}{\text{standard deviation}}$$



Negatively skewed



Positively skewed

2.2 DISPERSION

Measures Of Dispersion

The set of constants which would, in a concise way explain the "variability" or "spread" in a data is known as "measures of dispersion or variability". The average for two groups of the **same** number of measurements may be equal, but one group may be more variable than the other. For example, the set of five values 5, 6, 7, 8, 9 has the mean as 7; while another set of five values 1, 6, 4, 10, 14 also has the same mean 7. The second set has, obviously more variability than the first.

Usually four measures of dispersion or variability are defined. They are **(1)** Range, **(2)** Quartile deviation, **(3)** Average deviation and **(4)** Standard deviation.

Range: This is the difference between the two extreme observations in the data given. In the first set of values above, the Range R = 9 – 5 = 4 units, while in the second it is 14 – 1 = 13 units.

In a frequency distribution, R = (the largest x value) – (the smallest x value).

Obviously, this is not a very satisfactory measure except in cases where the number of observations are small and a quick calculation for the scatter is needed. It is used in statistical quality control studies rather widely.

Quartile Deviation: We know that the median bisects the distribution. If we divide the distribution, into four parts, we get what are called **quartiles**, Q_1, Q_2 = median and Q_3. The first quartile Q_1, would have 25 per cent of the values below it and the rest above it; the third quartile would have 75 per cent of values below it and the rest above it. The method of calculation of quartiles is similar to that of the median with slight variations.

The formulae are:

$$Q_1 = \ell + \frac{\frac{N}{4} - f_{Q_1}}{f} \times C;$$

$$Q_3 = \ell + \frac{3\frac{N}{4} - f_{Q_3}}{f} \times C$$

where we locate the Q_1 – class and Q_3 – class properly,

ℓ = lower limit of the quartile class

C = common factor

Quartile deviation Q.D. is defined as

$$\text{Q.D.} = \frac{1}{2}(Q_3 - Q_1)$$

Average deviation: If the average chosen is A (say), then the average deviation about A is A.D.(A), where

$$\text{A.D.(A)} = \frac{1}{n}\sum |(x_i - A)| \text{ for discrete data}$$

$$= \frac{1}{N}\sum f_i |x_i - A| \text{ for a frequency}$$

distribution.

Usually, we take either the median or the mean as the average.

Standard deviation (Root mean square deviation): The standard deviation σ is defined by

$$\sigma = \sqrt{\frac{1}{n}\sum (x_i - \bar{x})^2} \text{ for discrete data}$$

$$= \sqrt{\frac{1}{N}\sum f_i (x_i - \bar{x})^2} \text{ for a frequency}$$

distribution.

The square of the standard deviation, σ^2, is defined as the **variance** (V). Thus

$$V = \sigma^2 = \frac{1}{N}\sum f_i (x_i - \bar{x})^2$$

In recent literature, variance is used more extensively than the standard deviation.

Out of these measures, the last, σ, is widely used – as a companion to $\bar{x}$ on which it is based, when dealing with dispersion or scatter.

Formula for calculating V or σ

We know $V = \frac{1}{N}\sum f_i(x_i - \bar{x})^2$

Put $x_i = A + d_{iC}$

$$\text{Then } \bar{x} = A + \left(\frac{\sum f_i d_i}{N}\right)A + BC \text{ (say)}$$

$$\text{where } B = \left(\frac{\sum f_i d_i}{N}\right)$$

$$V = \frac{1}{N}\sum f_i[C\cdot(d_i - B)]^2$$

$$= \frac{C^2}{N}\cdot\sum f_i[d_i^2 - 2B\cdot d_i + B^2]$$

$$= \frac{C^2}{N}[\sum f_i d_i^2 - 2B(\sum f_i d_i) + B^2\sum f_i]$$

$$\text{But } \sum f_i = N \text{ and } \sum f_i d_i = NB$$

$$\therefore V = \frac{C^2}{N}[\sum f_i d_i^2 - 2B\cdot NB + B^2\cdot N$$

$$= \frac{C^2}{N}[\sum f_i\cdot d_i^2 - N\cdot B^2]$$

$$= \frac{C^2}{N}\left[\sum f_i\cdot d_i^2 - \frac{(\sum f_i d_i)^2}{N}\right]$$

$$\text{Thus, } V = \left[\frac{\sum f_i d_i^2}{N} - \left(\frac{\sum f_i d_i}{N}\right)^2\right]\times C^2$$

A few results of importance are:

1. The average deviation is least when taken from the median.
2. The standard deviation is not less than the average deviation from the mean.
3. For raw data $x_1, x_2, \ldots, x_n$

$$V = \frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2$$

$$\text{or } V = \frac{\sum x^2}{n} - (\bar{x})^2$$

4. If n_1, n_2 are the sizes of two groups; $\bar{x}_1, \bar{x}_2$ their means; and σ_1, σ_2 their standard deviations, then σ, the standard deviation of the combined group is determined from the following result.

$$(n_1+n_2)\sigma^2 = n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)$$

Where $d_1 = \bar{x} - \bar{x}_1$ and $d_2 = \bar{x} - \bar{x}_2$, $\bar{x}$ being the combined mean. This result can be extended to more than two groups.

Coefficients of dispersion: When two series of measurements have to be compared, we face the situation that the averages might also be different and the units in which the measurements are recorded may also be different. Hence, coefficients of dispersion are divided as given below. **Each of them is free from units of measurement and is a pure number.**

1. $\frac{\text{Range}}{A+B}$, where A is the largest and B is the smallest of the values.
2. $\frac{Q_3 - Q_1}{Q_3 + Q_1}$.
3. $\frac{\text{Average deviation about A}}{A}$
4. $\frac{\sigma}{\bar{x}} \times 100$, called the coefficient of variation (C.V.).

Thus coefficient of variation is a percentage.

Example: Compute the mean, median, mean deviation about mean, mean deviation about median, standard deviation, quartile deviation and coefficient of variation for the following data.

Marks	No. of Students
0 – 10	10
10 – 20	15
20 – 30	25
30 – 40	25
40 – 50	10
50 – 60	10
60 – 70	5

From the **Table** (see next page), we get

$$\text{Mean} = 35 + 10\left(-\frac{40}{100}\right) = 31$$

$$\text{Median} = \ell + \frac{\frac{N}{2} - f_m}{f}$$

$$= 30 + \frac{50 - 50}{25} = 30$$

$$\text{Mean deviation about Mean} = \frac{\sum fa_i}{100} = \frac{1300}{100} = 13$$

$$\text{Mean deviation about Median} = \frac{\sum fb_i}{100} = \frac{1300}{100} = 13$$

$$\text{Standard deviation} = \sqrt{\frac{\sum f_i d_i^2}{N} - \left(\frac{\sum f_i d_1}{N}\right)^2} \times 10$$

$$= \sqrt{\frac{270}{100} - (-0.4)^2} \times 10$$

$$= 10\sqrt{2.54} = 10 \times 1.594$$

$$= 15.94$$

To compute Quartile deviation, we find

$$Q_1 = \ell + \frac{\frac{N}{4} - f_{Q_1}}{f} \times C$$

$$= 20 + \frac{25 - 25}{25} \times 10 = 20$$

$$Q_3 = \ell + \frac{3\frac{N}{4} - f_{Q_3}}{f} \times C$$

$$= 40 + \frac{75 - 75}{10} \times 10 = 40$$

∴ quartile deviation

$$= \frac{1}{2}(Q_3 - Q_1) = \frac{1}{2}(40 - 20) = 10$$

$$\text{Coefficient of variation} = \frac{\text{S.D}}{\text{Mean}} \times 100$$

$$= \frac{15.94}{31} \times 100 = \mathbf{51.42\%}$$

2.3 ELEMENTS OF PROBABILITY

Experiment: An **experiment** is a process in which a certain work is repeated under the same conditions, the **outcomes** (or results) of which need not be the same. For instance, tossing a coin or rolling a die are experiments (A **die** is a cube with 1, 2, 3, 4, 5, 6 dots marked on its six faces).

Sample space: The set S of all possible outcomes of a given experiment is called the **sample space** for the experiment. An **outcome**, an element of S, is called a sample point. For the experiment of tossing a fair (or unbiased) coin, the sample space S = {H, T} where H, T refer to head, tail respectively.

Event: A subset of the sample space S is called an **event**. The set {a} consisting of a single sample point a in S is called an **elementary event**. Since the empty set ϕ and S are always subsets of S, they are also events. ϕ is called the **impossible event** or **null event**; S is called the **sure** or **certain event**.

Given two events A, B we can form new events using the set operations of union, intersection and complementation.

(i) $A \cup B$ is the event that occurs if and only if A occurs or B occurs (or both).

(ii) $A \cap B$ is the event that occurs if and only if both A and B occur.

(iii) A^c or $\overline{A}$, the complement of A, is the event that occurs if and only if A does not occur.

Mutually Exclusive Events: Two events A and B are said to be **mutually exclusive** if and only if they cannot occur simultaneously, that is, if $A \cap B = \phi$.

Three or more events are called mutually exclusive, if every two of them are mutually exclusive.

Example: Consider the experiment of rolling a fair die. Then S = {1, 2, 3, 4, 5, 6}.

Let A = event that an odd number appears = {1, 3, 5}

B = event that an even number appears = {2, 4, 6}

C = event that a prime number appears = {2, 3, 5}

Then A and B are mutually exclusive events since $A \cap B = \phi$ whereas A and C are not mutually exclusive as $A \cap C = \{3, 5\} \neq \phi$.

Marks	Mid-Mark (x_i)	$d_i = \frac{x_i - 35}{10}$	f (frequency)	cum f	fd_i	fd_i^2	Abs. val. of deviation from Mean (a_i) (31)	Abs. val. of deviation from Median (b_i) (30)	Total deviation from Mean fa_i	Total deviation from Median fb_i
0 – 10	5	– 3	10	10	– 30	90	26	25	260	250
10 – 20	15	– 2	15	25	– 30	60	16	15	240	225
20 – 30	25	– 1	25	50	– 25	25	6	5	150	125
30 – 40	35	0	25	75	0	0	4	5	100	125
40 – 50	45	1	10	85	10	10	14	15	140	150
50 – 60	55	2	10	95	20	40	24	25	240	250
60 – 70	65	3	5	100	15	45	34	35	170	175
Total			100		– 40	270			1300	1300

Problem: Find the total number of possible events that can occur for an experiment.

Solution: Let the sample space S consist of n sample points. Then the total number of possible events = the total number of subsets of S = $|P(S)|$, where P(S) is the power set of S = 2^n.

Problem: A, B, C are three events. Draw Venn diagrams for the following events:

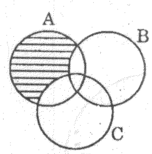
(i) Only A occurs **(ii)** A and B occur but C does not occur

(iii) None occurs **(iv)** Simultaneous occurrence of A, B, C

(v) A, B and C are mutually exclusive.

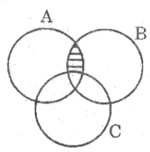
Solution

(i)



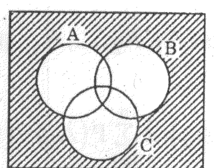
$A \cap \overline{B} \cap \overline{C}$

(ii)



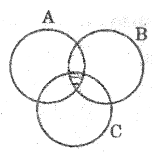
$A \cap B \cap \overline{C}$

(iii)



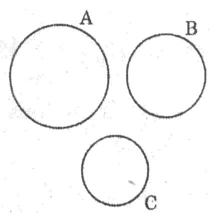
$\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$

(iv)



$A \cap B \cap C$

(v)



$A \cap B = \phi, B \cap C = \phi, C \cap A = \phi$

Problem: In an experiment, a fair coin is tossed 4 times. Describe the sample space.

Solution: The sample space S consists of 16 (= 2^4) sample points.

S = {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTTH, TTHH, THTH, HTHT, THHT, HTTT, THTT, TTHT, TTTH, TTTT}

Finite sample space: A sample space S is said to be a **finite sample space**, if S is a finite set.

Finite probability space: Let $S = \{a_1, a_2, \ldots, a_n\}$ be a finite sample space. S is said to be a **finite probability space** or **probability model**, if each sample point a_i in S we can assign a real number p_i, called the **probability of a_i** such that

(i) $p_i \geq 0$, for each i

(ii) $p_1 + p_2 + \ldots + p_n = 1$

If A is an event, then the **probability of A**, denoted by P(A), is the sum of the probabilities of the sample points in A. For the elementary event $\{a_i\}$, we write $P(a_i)$ instead of $P(\{a_i\})$.

Equiprobable space: A finite probability space S in which each sample point has the same probability is called an **equiprobable space**.

If E is an event, then

$$P(E) = \frac{\text{number of elements in E}}{\text{number of elements in S}}$$

$$= \frac{|E|}{|S|}$$

$$= \frac{\text{number of outcomes favourable to E}}{\text{total number of possible outcomes}}$$

Note 1: We use the word "at random" only when dealing with an equiprobable space. By the statement, "a ball is drawn at random from a bag containing 10 balls" we mean that each ball in the bag has the same probability of being chosen.

Note 2: If p = probability of happening of an event E = P(E) and q = probability of not happening of $E = P(\overline{E})$ then $p + q = 1$,

$$\text{for } q = P(\overline{E}) = \frac{|\overline{E}|}{|S|} = \frac{|S| - |E|}{|S|}$$

$$= 1 - \frac{|E|}{|S|} = 1 - p$$

$$\therefore p + q = 1$$

Note 3: Properties of probability: Let *C* be the family of all events in a finite probability space S. Then the probability function P defined on *C* satisfies

(P_1) $0 \le P(A) \le 1$, for every event A

(P_2) $P(S) = 1$

(P_3) If A, B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

Problem: For any two events A, B prove that

(a) $P(\overline{A}) = 1 - P(A)$

(b) $P(\phi) = 0$

(c) $P(A - B) = P(A) - P(A \cap B)$

i.e., $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

(d) $A \subseteq B \Rightarrow P(A) \le P(B)$

(e) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(Addition Principle)

Solution

(a) $A \cup \overline{A} = S$ and $(A \cap \overline{A}) = \phi$

$\therefore$ by (P_2) and (P_3),

$1 = P(S) = P(A \cup \overline{A}) = P(A) + P(\overline{A})$

or $P(\overline{A}) = 1 - P(A)$

(b) $A \cup \phi = A$ and $A \cap \phi = \phi$

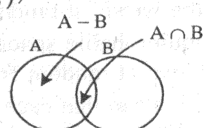
$\therefore$ by (P_3), $P(A) = P(A \cup \phi) = P(A) + P(\phi)$

or $P(\phi) = 0$

(c) $(A - B) \cup (A \cap B) = A$ and

$(A - B) \cap (A \cap B) = \phi$

$\therefore$ by (P_3),



$P(A) = P[(A - B) \cup (A \cap B)]$

$= P(A - B) + P(A \cap B)$

or $P(A - B) = P(A) - P(A \cap B)$

(d) $A \subseteq B \Rightarrow A \cup (B - A) = B$ and

$A \cap (B - A) = \phi$

$\therefore$ by (P_3), $P(B) = P[A \cup (B - A)]$

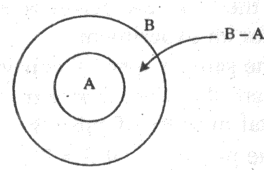
$= P(A) + P(B - A)$

By (P_1), $P(B - A) \ge 0$

$\therefore P(A) \le P(B)$

(e) $A \cup B = (A - B) \cup B$ and

$(A - B) \cap B = \phi$



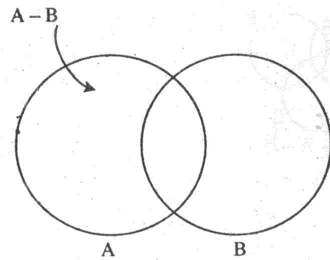
$\therefore$ by (P_3),

$P(A \cup B) = P[(A - B) \cup B]$

$= P(A - B) + P(B)$

$= P(A) - P(A \cap B) + P(B)$ by (c)

$= P(A) + P(B) - P(A \cap B)$.



Problem: For any n events $A_1, A_2, \ldots, A_n$ prove that

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) \ge \sum_{i=1}^{n} P(A_i) - (n-1)$$

Solution: Proof is by induction on n.

Let P_n be the statement.

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) \ge \sum_{i=1}^{n} P(A_i) - (n-1)$$

When n = 1 , P (1) is obvious since both sides become equal to $P(A_1)$.

When $n = 2$, LHS $= P(A_1 \cap A_2)$

$= P(A_1) + P(A_2) - P(A_1 \cup A_2)$

(by addition principle)

$\ge P(A_1) + P(A_2) - 1$

$(\because P(A_1 \cup A_2) \le 1$

$= P(A_1) + P(A_2) - (2 - 1)$ ... (1)

$\therefore P(2)$ is true.

Assume that P(k) is true. That is,

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) \geq \sum_{i=1}^{k} P(A_i) - (k-1) \quad \ldots(2)$$

Now, $P(A_1 \cap A_2 \cap \ldots\ldots \cap A_{k+1})$

$$= P\left[(A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1}\right]$$

$$\geq P\left[(A_1 \cap A_2 \cap \cdots \cap A_k) + P(A_{k+1}) - (2-1)\right]$$

$$\geq \sum_{i=1}^{k} P(A_i) - (k-1) + P(A_{k+1}) - 1, \text{ by (2)}$$

$$= \sum_{i=1}^{k+1} P(A_i) - ((k+1)-1)$$

This shows that P(k + 1) is true.

Thus, by induction, P(n) is true for all n.

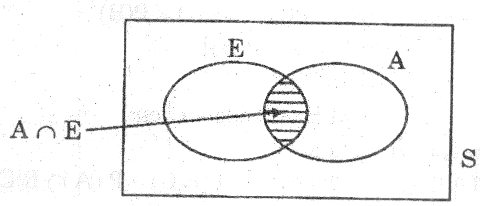
Problem: Show that

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq \sum_{i=1}^{n} P(A_i)$$

Solution: Proof is by induction on n (left as an exercise).

Conditional Probability: Let E be an event in a sample space S with **P(E) > 0**. Suppose an event A occurs after the occurrence of E. Then the **conditional probability of A given E**, denoted by

P(A|E), is given by $P(A|E) = \dfrac{P(A \cap E)}{P(E)}$.



Conditional Event

We can say that P(A|E) represents the probability of A with respect to the reduced sample space E. When S is an equiprobable space,

$$P(A \cap E) = \frac{|A \cap E|}{|S|}, \; P(E) = \frac{|E|}{|S|}.$$

$$\text{So, } P(A|E) = \frac{P|A \cap E|}{P|E|} = \frac{|A \cap E|}{|E|}$$

$$= \frac{\text{number of elements in A in E}}{\text{number of elements} \cap \text{E}}$$

Note: $P(A \mid E) \cdot P(E) = P(A \cap E)$ is known as Multiplication Theorem for conditional probability.

Problem : Let A and B be events with

$$P(A) = \frac{3}{8}, \; P(B) = \frac{5}{8} \text{ and } P(A \cup B) = \frac{3}{4}.$$

Find P(A|B) and P(B|A).

Solution

$$\frac{3}{4} = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{8} + \frac{5}{8} - P(A \cap B)$$

$$\therefore P(A \cap B) = \frac{1}{4}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{2}{5} \text{ and}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{2}{3}$$

Problem: Suppose a pair of fair dice is rolled. If the sum is 6, what is the probability that one die shows a 2?

Solution

The sample space S consists of $6 \times 6 = 36$ sample points ((1, 1), (1, 2), … (6, 5), (6, 6)).

E = sum is 6 = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}

A = 2 appears on at least one die

= {(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (1, 2), (3, 2), (4, 2), (5, 2), (6, 2)}.

Also $A \cap E$ = {(2, 4), (4, 2)}

$$\text{Thus } P(E) = \frac{5}{36}, \; P(A) = \frac{11}{36}, \; P(A \cap E) = \frac{2}{36}$$

$$\therefore \text{ required probability} = P(A|E) = \frac{P(A \cap E)}{P(E)}$$

$$= \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$

Independent Events: Events A and B in the space S are said to be **independent** if the occurrence of one of them does not influence the occurrence of

the other; that is, B is independent of A if $P(B|A) = P(B)$.

$\therefore$ by using this in the multiplication theorem for conditional probability, we get

$$P(A \cap B) = P(A) \cdot P(B|A) = P(A)\,P(B)$$

Thus events A and B are **independent** if

$$P(A \cap B) = P(A) \cdot P(B)$$

Problem: Two research scholars A and B work **independently** on a problem. The probability that A will solve it is $\frac{3}{5}$ and the probability that B will solve it is $\frac{3}{4}$. What is the probability that the problem will be solved?

Solution: Given P(A) = probability of A solving the problem = $\frac{3}{5}$ and P(B) = $\frac{3}{4}$. Since A and B work independently the two events are independent.

$$\therefore P(A \cap B) = P(A) \times P(B) = \frac{3}{5} \times \frac{3}{4} = \frac{9}{20}$$

Probability that the problem will be solved

$$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{5} + \frac{3}{4} - \frac{9}{20} = \frac{9}{10}.$$

Problem : A box contains 3 white balls and 2 green balls. Three persons A, B, C in that order draw a ball which is not replaced. The first to draw a white ball is the winner. Find their respective probabilities of success.

Solution: Probability that A draws a white ball $= \frac{3}{5}$

Probability that A fails to draw a white ball

$$= 1 - \frac{3}{5} = \frac{2}{5}$$

When A fails to draw a white ball, B gets his chance.

$\therefore$ probability for B drawing a white ball

= probability that A fails to draw a white ball $\times$ probability that B draws a white ball

$$= \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$$

$\therefore$ probability that B fails to draw a white ball

$$= 1 - \frac{3}{10} = \frac{7}{10}$$

When B fails, C gets his chance.

$\therefore$ probability that C draws a white ball

= probability that A fails $\times$ probability that B fails $\times$ probability that C draws a white ball

$$= \frac{2}{5} \times \frac{7}{10} \times \frac{3}{3} = \frac{7}{25}$$

Thus the respective probabilities of success are

$$\frac{3}{5}, \frac{3}{10}, \frac{7}{25}.$$

Problem: If A and B are independent events prove that

(a) A and $\overline{B}$ are independent

(b) $\overline{A}$ and $\overline{B}$ are independent.

Solution: A, B independent

$$\Rightarrow P(A \cap B) = P(A) \times P(B) \qquad \ldots (1)$$

(a) $P(A \cap \overline{B}) = P(A - B) = P(A) - P(A \cap B)$

$= P(A) - P(A) \times P(B)$ (by (1))

$= P(A)\,[1 - P(B)]$

$= P(A) \times P(\overline{B})$

$\therefore$ A and $\overline{B}$ are independent.

(b) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$ (by addition principle)

$= 1 - [P(A) + P(B) - P(A) \times P(B)]$ (by (1))

$= 1 - P(A) - P(B) + P(A) \times P(B)$

$= [1 - P(A)]\,[1 - P(B)]$

$= P(\overline{A}) \times P(\overline{B})$

$\therefore$ $\overline{A}$ and $\overline{B}$ are independent.

Problem: Prove that

$$P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$$

Solution:

$P((A \cup B) \cap C) = P((A \cap C) \cup (B \cap C))$

$= P(X \cup Y)$

where $X = A \cap C$, $Y = B \cap C$

$= P(X) + P(Y) - P(X \cap Y)$

$= P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))$

$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

Divide both sides by P(C) ($\because$ P(C) > 0). Then

$$\frac{P((A\cup B)\cap C)}{P(C)}=\frac{P(A\cap C)}{P(C)}+\frac{P(B\cap C)}{P(C)}-\frac{P(A\cap B\cap C)}{P(C)}$$

or $P(A \cup B \mid C) = P(A|C) + P(B|C) - P(A \cap B|C)$

Problem: Show that

$$P(A\cap\bar{B}|C)+P(A\cap B|C)=P(A|C)$$

Solution

$$(A\cap\bar{B}\cap C)\cap(A\cap B\cap C)=A\cap C\cap(B\cap\bar{B})$$
$$=A\cap C\cap\phi=\phi \quad ...(1)$$

i.e., $A\cap\bar{B}\cap C$ and $A\cap B\cap C$ are mutually exclusive

Now, $P(A\cap\bar{B}|C)+P(A\cap B|C)$

$$=\frac{P(A\cap\bar{B}\cap C)}{P(C)}+\frac{P(A\cap B\cap C)}{P(C)}$$

$$=\frac{P(A\cap\bar{B}\cap C)+P(A\cap B\cap C)}{P(C)}$$

$$=\frac{P[(A\cap\bar{B}\cap C)\cup(A\cap B\cap C)]}{P(C)}$$

on using (1)

$$=\frac{P[(A\cap C)\cap(\bar{B}\cup B)]}{P(C)}$$

$$=\frac{P[(A\cap C)\cap S]}{P(C)}=\frac{P(A\cap C)}{P(C)}$$
$$=P(A|C)$$

Mutual Independence: Let $\{E_k : k \text{ is a positive integer}\}$ be a collection of events. These events are said to be **mutually independent** if for each finite non-empty subset $\{E_1, E_2, \ldots\ldots E_n\}$

$$P(E_1\cap E_2\cap\ldots\cap E_n)=P(E_1)\,P(E_2)\ldots P(E_n)$$

Pairwise independence: Let $\{E_k : k \text{ is a positive integer}\}$ be a collection of events. These events are said to be **pairwise independent** if

$$P(E_i\cap E_j)=P(E_i)\cdot P(E_j) \text{ for all } i\neq j.$$

Problem: If A, B, C are mutually independent, then show that $A \cup B$ and C are independent.

Solution: A, B, C are mutually independent.

$\Rightarrow P(A \cap B) = P(A)\,P(B)$

$P(B \cap C) = P(B)\,P(C)$

$P(A \cap C) = P(A)\,P(C)$ and

$P(A \cap B \cap C) = P(A)\,P(B)\,P(C) \quad ...(1)$

Now, $P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$

$= P(A \cap C) + P(B \cap C) - P(A \cap C \cap B \cap C)$

$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

$= P(A)\,P(C) + P(B)\,P(C) - P(A)\,P(B)\,P(C)$

(on using (1))

$= P(C)\,[P(A) + P(B) - P(A)\,P(B)]$

$= P(C)\,[P(A) + P(B) - P(A \cap B)]$ by (1)

$= P(C)\,P(A \cup B)$

$\therefore A \cup B$ and C are independent.

Problem: A, B, C are random events in a sample space. If A, B, C are pairwise independent and A is independent of $B \cup C$ show that A, B and C are mutually independent.

Solution: A, B, C are pairwise independent.

$$\left.\begin{aligned}\therefore P(A \cap B) &= P(A)\,P(B)\\ P(B \cap C) &= P(B)\,P(C)\\ P(A \cap C) &= P(A)\,P(C)\end{aligned}\right\} \quad ...(1)$$

A is independent of $B \cup C$

$\therefore P[A \cap (B \cup C)] = P(A) \cdot P(B \cup C) \quad ...(2)$

To prove A, B, C are mutually independent. In view of results (1) it is enough to prove that

$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$.

Now $P[(A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$

$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C)$

$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$

i.e., $P(A)\,P(B \cup C) = P(A)\,P(B) + P(A)\,P(C) - P(A \cap B \cap C)$ (by (1) and (2))

or $P(A \cap B \cap C) = P(A)\,[P(B) + P(C) - P(B \cup C)]$

$= P(A)\,[P(B) + P(C) - P(B) - P(C) + P(B \cap C)]$

$= P(A)\,P(B \cap C) = P(A)\,P(B)\,P(C)$

Thus A, B, C are mutually independent.

Possible Event: An event E is said to be a **possible event**, if $P(E) > 0$.

Problem : Prove or disprove:

(a) Possible independent events are mutually exclusive.

(b) Possible mutually exclusive events are independent.

Solution: We disprove both statements.

(a) Let A, B be possible independent events. Then $P(A) > 0$, $P(B) > 0$ and $P(A \cap B) = P(A)\,P(B)$.
$\therefore P(A \cap B) \neq 0$ and so $A \cap B \neq \phi$ i.e., A, B are not mutually exclusive.

(b) Let A, B be possible mutually exclusive events.
Then $P(A) > 0$, $P(B) > 0$ and $A \cap B = \phi$.
$\therefore P(A \cap B) = P(\phi) = 0 \neq P(A)\,P(B)$. Thus A, B are not independent.

Problem: Box A contains three balls with colours red, green and blue. Box B contains five balls with colours red, yellow, blue, white and brown. A box is chosen and a ball is drawn. What is the probability that the ball drawn is **(i)** brown **(ii)** green or blue and **(iii)** red or blue?

Solution: The event of choosing a ball occurs only after choosing a box.
Probability of choosing any one box $= \frac{1}{2}$

(i) A brown ball can be drawn only from box B.
$\therefore$ P (brown) = Probability of choosing box B $\times$ probability of drawing a brown ball

$$= \frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$$

(ii) Box A contains green and blue balls whereas Box B contains no green ball
$\therefore$ P (green or blue)
= P (green or blue from box A) + P (green or blue from box B)

$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{5} = \frac{13}{30}$$

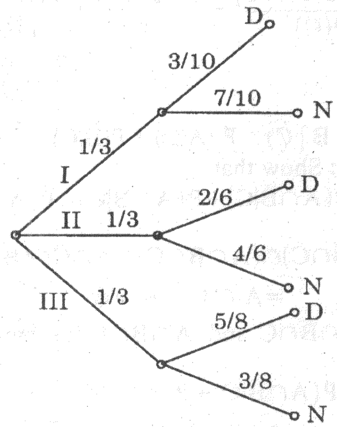
(iii) Both boxes contain red and blue balls.

$$\therefore \text{P(red or blue)} = \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{2}{5} = \frac{8}{15}$$

Problem : Box I has 10 light bulbs of which 3 are defective; Box II has 6 light bulbs of which 2 are defective; Box III has 8 light bulbs of which 5 are defective. A box is chosen at random and a bulb is drawn at random. **(i)** What is the probability that the bulb drawn is defective? **(ii)** What is the probability that the defective bulb was from Box I?

Solution: We solve the problem using a tree diagram; D and N stand for defective and non-defective cases. The various probabilities are marked on the branches of the tree.

By following the branches,



(i) P(D) = Probability of a defective bulb

$$= \frac{1}{3} \times \frac{3}{10} + \frac{1}{3} \times \frac{2}{6} + \frac{1}{3} \times \frac{5}{8}$$

$$= \frac{1}{3} \times \frac{302}{240} = \frac{302}{720}$$

(ii) Probability that the defective bulb was from box I
= P(A|D) (where A is the event of choosing box I)

$$= \frac{P(A \cap D)}{P(D)}$$

$$= \frac{\frac{1}{3} \times \frac{3}{10}}{\frac{302}{720}}$$

$[\because A \cap D =$ defective from box I

$$\Rightarrow P(A \cap D) = \frac{1}{3} \times \frac{3}{10}\,]$$

$$= \frac{72}{302}$$

Problem: Let p be the probability that a man aged y years will meet with an accident in a year. What is the probability that a man among n men all aged y years will meet with an accident first?

Solution: Probability that a man aged y years will not meet with an accident $= 1 - p$. $\therefore$ given n men,
P(none meets with an accident)

$$= \underbrace{(1-p)(1-p)\,.....\,(1-p)}_{\text{n times}}$$
$$= (1-p)^n$$

$\therefore$ P(atleast one man meets with an accident)

$$= 1-(1-p)^n$$

So, P (atleast one man meets with an accident | a person is chosen)

$$= \frac{1}{n} \times [1-(1-p)^n]$$

Problem: A box contains 4 white, 3 blue and 5 green balls. Four balls are chosen. What is the probability that all three colours are represented?

Solution: Total number of balls in the box is 12.

$\therefore$ the total number of ways in which 4 balls can be chosen

$$= {}^{12}C_4 = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495$$

Each colour will be represented in the following mutually exclusive ways:

	White	Blue	Green
(i)	2	1	1
(ii)	1	2	1
(iii)	1	1	2

$\therefore$ number of ways of drawing four balls in the above fashion

$$= {}^4C_2 \times {}^3C_1 \times {}^5C_1 + {}^4C_1 \times {}^3C_2 \times {}^5C_1$$
$$+ {}^4C_1 \times {}^3C_1 \times {}^5C_2 = 90+60+120 = 270$$

So, the required probability $= \frac{270}{495}$

Problem: Three people are present in a room. What is the probability that the birth dates of at least two will fall on the same day of the month? (Assume 30 days for a month).

Solution: Since the birth date of any person can fall on anyone of the 30 days the exhaustive number of cases for the birth dates of 3 persons

$$= 30 \times 30 \times 30 = 30^3.$$

If the birth dates of all the 3 persons fall on different dates, then the number of favourable cases is $30\,(30-1)\,(30-2) = 30 \times 29 \times 28$. For, the birth date of the first person can fall on anyone of 30 days, the birth date of the second person on anyone of the remaining 29 days and the birth date of the third person on anyone of the remaining 28 days.

$\therefore$ p = probability that birth dates of the 3 persons are different

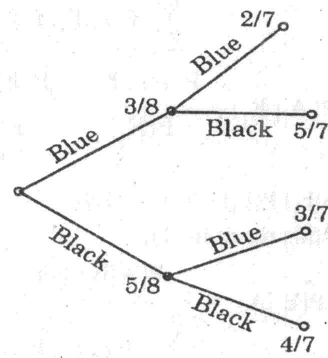
$$= \frac{30 \times 29 \times 28}{30^3} = \frac{29 \times 28}{30^2}$$
$$= \left(1 - \frac{1}{30}\right)\left(1 - \frac{2}{30}\right)$$

So, the required probability that at least two persons will have the same birth date

$$= 1 - p = 1\left(1 - \frac{1}{30}\right)\left(1 - \frac{2}{30}\right).$$

Problem: A box has 3 blue and 5 black colour balls. One ball is drawn and kept aside without noting its colour. Then another ball is drawn. Find the probability that it is blue.

Solution



$$\text{Required probability} = \frac{3}{8} \times \frac{2}{7} + \frac{5}{8} \times \frac{3}{7}$$
$$= \frac{3}{8}$$

Law of Total Probability: Let $B_1, B_2, ..., B_n$ be mutually exclusive and let an event A occur only if anyone of B_i occurs. Then

$$P(A) = \sum_{i=1}^{n} P(A|B_i)\, P(B_i)$$

Baye's Theorem: Let $E_1, E_2, ..., E_n$ be mutually exclusive events such that $P(E_i) > 0$ for each i.

Then for any event $A \subseteq \bigcup_{1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i|A) = \frac{P(A|E_i)\,P(E_i)}{\sum_{i=1}^{n} P(A|E_i)\,P(E_i)}, \; i = 1, 2, \ldots\ldots, n$$

Proof: We know

$$P(E_i|A) = \frac{P(E_i \cap A)}{P(A)} \qquad \ldots(1)$$

By the law of total probability

$$P(A) = \sum_{i=1}^{n} P(A|E_i)\,P(E_i) \qquad \ldots(2)$$

$\therefore$ from (1) and (2),

$$P(E_i|A) = \frac{P(E_i \cap A)}{\sum_{i=1}^{n} P(A|E_i)\,P(E_i)} \qquad \ldots(3)$$

Also $P(A \mid E_i) = \dfrac{P(A \cap E_i)}{P(E_i)} = \dfrac{P(E_i \cap A)}{P(E_i)}$

$\therefore$ $P(A|E_i)\,P(E_i) = P(E_i \cap A)$

Using this result in (1),

$$P(E_i|A) = \frac{P(A|E_i)\,P(E_i)}{\sum_{i=1}^{n} P(A|E_i)\,P(E_i)}.$$

Problem: Three urns A, B and C have 1 white, 2 black, 3 red balls; 2 white, 1 black, 1 red ball and 4 white, 5 black, 3 red balls respectively. One urn is chosen at random and two balls are drawn. They happen to be white and red balls. What is the probability that they came from urn B?

Solution: Let E_1, E_2 and E_3 be the events of choosing urns A, B and C respectively. Then

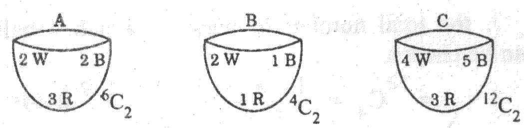
$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3} \qquad \ldots(1)$$

Let X be the event of choosing two balls, white and red. To find $P(E_2|X)$.

By Baye's Theorem,

$$P(E_2|X) = \frac{P(X|E_2)\,P(E_2)}{P(X|E_1)\,P(E_1) + P(X|E_2)\,P(E_2) + P(X|E_3)\,P(E_3)}$$

$$= \frac{P(X|E_2)}{P(X|E_1) + P(X|E_2) + P(X|E_3)} \text{(on using(1))} \qquad \ldots(2)$$



$$\text{Now } P(X|E_1) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1}{5}$$

$$P(X|E_2) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{1}{3} \text{ and}$$

$$P(X|E_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{2}{11}$$

$$\therefore \text{ by (2), } P(E_2|X) = \frac{\frac{1}{3}}{\frac{1}{5} + \frac{1}{3} + \frac{2}{11}} = \frac{55}{118}$$

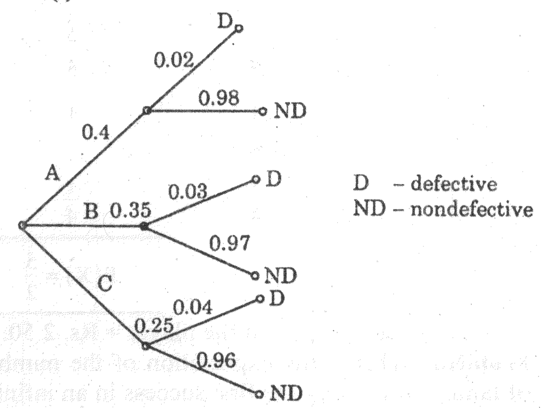
Problem: Three machines A, B, C produce respectively 40%, 35%, 25% of the total number of items of a factory. The percentages of defective output of these machines are 2%, 3%, 4%.

(i) If an item is selected at random, find the probability that the item is defective.

(ii) Suppose an item is selected at random and is found to be defective. Find the probability that the item was produced by machine A.

Solution: Let X be the event of an item being defective.

(i)



$\therefore P(X) = 0.4 \times 0.02 + 0.35 \times 0.03 + 0.25 \times 0.04$

$= 0.0285.$

(ii) By Baye's Theorem,

$$P(A|X) = \frac{0.4 \times 0.02}{0.0285} = \frac{16}{57}.$$

Bernouilli Trials: Independent repeated trials of an experiment with two outcomes only are called Bernouilli trials, named after Jacob Bernouilli. Call one of the outcomes success and the other outcome failure.

Let p = probability of success in a Bernouilli trial

q = probability of failure = 1 – p.

A binomial experiment consisting of a fixed number n of trials is denoted by B(n, p). The probability of r success in the experiment B(n, p) is given by

$$P(r) = {}^nC_r p^r q^{n-r}.$$

The function P(r) for r = 0, 1, 2, ..., n for B(n, p) is called the **binomial distribution**.

Problem: Assuming that the ratio of male children is $\frac{1}{2}$, find the probability that in a family of 7 children

(i) all children will be of the same sex;

(ii) the four oldest children will be girls and the youngest will be boys.

Solution

(i) All the 7 children can be either boys or girls. Number of favourable ways = 2.

Total number of ways = 2^7

$\therefore$ required probability = $\frac{2}{2^7} = \frac{1}{64}$

(ii) Out of 7 children 4 can be girls in ${}^7C_4 = 35$ ways,

$\therefore$ probability that the four oldest children will be girls = $\frac{1}{35}$

Problem: If four coins are tossed, find the chance that there should be two heads and two tails.

Solution: p = probability of a head = $\frac{1}{2}$;

q = probability of a tail = 1 – p = $\frac{1}{2}$

$\therefore$ probability of getting 2 heads (and 2 tails) when four coins are tossed

$$= P(2) = {}^4C_2 p^2 q^{4-2} = 6 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{8}$$

Random Variable: A **random variable** X is a rule which assigns a numerical value to each outcome in a sample space S. If R_X is the set of numbers assigned by X, then R_X is called the **range space**.

Let X be a random variable. Then

(a) the function $F(x) = P(X \le x)$ is called the **distribution function** of X.

(b) **Mean** or **Expectation of X** = μ = E(X)

$$= X(a_1)P(a_1) + \dots\dots\dots + X(a_n)P(a_n)$$

$$= \sum_{i=1}^{n} X(a_i)\,P(a_i),$$ where S is the probability space $\{a_1, a_2, \dots\dots, a_n\}$

(c) **Variance of X** = Var(X) = σ^2 = $E(X^2) - [E(X)]^2$ and **standard deviation of X** = σ.

Problem: Find the expectation of the number of points when a fair die is rolled.

Solution: Let X be the random variable showing the number of points.

Then X = 1, 2, 3, 4, 5, 6

a_i	$P(X = a_i) = P(a_i)$	Product
1	$\frac{1}{6}$	$\frac{1}{6}$
2	$\frac{1}{6}$	$\frac{2}{6}$
3	$\frac{1}{6}$	$\frac{3}{6}$
4	$\frac{1}{6}$	$\frac{4}{6}$
5	$\frac{1}{6}$	$\frac{5}{6}$
6	$\frac{1}{6}$	$\frac{6}{6}$
		$E(X) = \frac{21}{6} = \frac{7}{2}$

$\therefore$ expectation $= \frac{7}{2}$

Problem: A player tosses 2 fair coins. He wins Rs. 5 if 2 heads occur and Rs. 2 if 1 head occurs and Re. 1 if no heads occur. Find his expected winnings.

Solution: S = {HH, HT, TH, TT}

$$P(\text{2 heads}) = P(HH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(\text{1 head}) = P(HT) + P(TH)$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

$$P(\text{no head}) = P(TT) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Let X be the random variable showing earning. Then X = 5, 2, 1

a_i	$P(a_i)$	Product
5	$\frac{1}{4}$	$\frac{5}{4}$
2	$\frac{1}{2}$	1
1	$\frac{1}{4}$	$\frac{1}{4}$
		$E(X) = \frac{5}{2}$

$\therefore$ expected earnings of the player = Rs. 2.50.

Problem: What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability of success p in each trial?

Solution: p = probability of success

$\therefore$ q = probability of failure = 1 − p

Let X be the random variable representing the number of failures preceding the first success. Since an infinite series of independent trials are conducted, X = 0, 1, 2, ...

P(X = x) = probability that there are x failures preceding the first success

$$= \underbrace{q \times q \times \ldots\ldots \times q}_{x \text{ times}} \times p = q^x p$$

$$\therefore E(X) = \sum_{x=0}^{\infty} x p(X=x)$$

$$= \sum_{x=0}^{\infty} x q^x p$$

$$= 1qp + 2q^2p + 3q^3p + \ldots\ldots$$

$$= qp(1 + 2q + 3q^2 + \ldots\ldots)$$

$$= qp(1-q)^{-2} [\because (1-q)^{-2} = 1 + 2q + 3q^2 + \ldots, \text{ by Binomial Theorem}]$$

$$= \frac{qp}{(1-q)^2} = \frac{qp}{p^2} = \frac{q}{p}$$

Problem: The Bernouilli probability law with parameter p in which $0 \le p \le 1$ is defined by a random variable X taking the value 1 with probability p and the value 0 with probability q. Find the mean and variance of X.

Solution

a_i	$P(a_i)$	Product
1	p	p
0	q	0
		E (X) = p

$\therefore$ mean = E(X) = p.

Now, to find Var(X):

a_i	a_i^2	$P(a_i)$	Product $a_i^2 \times P(a_i)$
1	1	p	p
0	0	q	0
			$E(X^2) = p$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2 = p - p^2$$
$$= p(1-p) = pq.$$

Note: Suppose a value x_i is repeated f_i times. Then f_i is called the **frequency** or **absolute frequency**.

Let N = sum of all frequencies f_i.

Then $\frac{f_i}{N}$ is called the **relative frequency**,

i = 1, 2, ..., n.

Sum of all relative frequencies

$$= \frac{f_1}{N} + \frac{f_2}{N} + \ldots\ldots\ldots + \frac{f_n}{N}$$
$$= \frac{f_1 + f_2 + \ldots\ldots\ldots + f_n}{N}$$
$$= \frac{N}{N} = 1$$

Problem: Find the mean, sample variance and standard deviation of the sample 1, 3, 6, 5, 10.

Solution: Mean, $\bar{x} = \frac{1+3+6+5+10}{5} = 5$

Sample variance $= \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

(formula)

$$= \frac{1}{4} \sum_{i=1}^{5} (x_i - 5)^2 (\because n = 5)$$
$$= \frac{1}{4} [(1-5)^2 + (3-5)^2 + (6-5)^2 + (5-5)^2 + (10-5)^2]$$
$$= \frac{1}{4} [4^2 + 2^2 + 1^2 + 0^2 + 5^2] = 11.5$$

Standard deviation, $\sigma = \sqrt{\mathbf{11.5}}$

OBJECTIVE QUESTIONS

1. The measure of central tendency based on all observations is
   (A) arithmetic mean **(B)** geometric mean
   (C) median **(D)** mode
2. The algebraic sum of the deviations of a set of values from their arithmetic mean is
   (A) positive **(B)** zero
   (C) negative **(D)** indeterminate
3. The probability of drawing one white ball from a box having 3 black, 4 white and 2 red balls is
   (A) $\frac{1}{9}$ **(B)** $\frac{2}{9}$
   (C) $\frac{3}{9}$ **(D)** $\frac{4}{9}$
4. When a fair coin is tossed four times the sample space consists of n sample points where n is
   (A) 2 **(B)** 4
   (C) 8 **(D)** 16
5. Which of the following statements is **true**?
   (A) Geometric mean ≥ Harmonic mean ≥ Arithmetic mean
   (B) Harmonic mean ≥ Geometric mean ≥ Arithmetic mean
   (C) Geometric mean ≥ Arithmetic mean ≥ Harmonic mean
   (D) Arithmetic mean ≥ Geometric mean ≥ Harmonic mean
6. Which of the following statements is **false**?
   (A) Arithmetic mean ≥ Geometric mean
   (B) Geometric mean ≥ Harmonic mean
   (C) Arithmetic mean ≥ Harmonic mean
   (D) Geometric mean ≥ Harmonic mean ≥ Arithmetic mean

7. The most stable measure of central tendency is
(A) mode (B) median
(C) geometric mean (D) mean

8. To find the most selling shoe size the suitable measure of central tendency is
(A) mode (B) median
(C) mean (D) none of the above

9. If A and B are mutually exclusive events, then
(A) $P(A) \leq P(B)$ (B) $P(A) \leq P(\overline{B})$
(C) $P(A) \geq P(B)$ (D) $P(A) \geq P(\overline{B})$

10. If A contains B, then
(A) $P(A) < P(A \cup B)$
(B) $P(A) > P(A \cup B)$
(C) $P(A) = P(A \cup B)$
(D) $P(A) \neq P(A \cup B)$

11. A number is chosen at random among the first 10 natural numbers. Then the probability that the number is prime and even equals
(A) 0 (B) 1
(C) $\frac{1}{10}$ (D) none of the above

12. If A and B are any two arbitrary events, then
(A) $P(A \cup B) = P(A) + P(B)$
(B) $P(A \cup B) \leq P(A) + P(B)$
(C) $P(A \cup B) = P(A)$
(D) $P(A \cup B) = P(B)$

13. Skewness is
(A) a measure of central tendency
(B) lack of symmetry
(C) a measure of dispersion
(D) none of the above

14. If $P(A) = 0.24$, $P(B) = 0.4$ and $P(A \cup B) = 0.2$, then
(A) A and B are mutually exclusive
(B) A and B are independent
(C) $P(A \cap B) = 0.44$
(D) $P(A) > P(\overline{B})$

15. If $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cup B) = 0.85$, then
(A) A and B are independent
(B) A and B are mutually exclusive
(C) $P(A|B) = \frac{1}{12}$
(D) $P(A \cap B) = 0.5$

16. The probability that it will rain today is 0.5. The probability that it will rain tomorrow is 0.6. The probability that it will rain either today or tomorrow is 0.7. Then the probability that it will rain today and tomorrow is
(A) 0.3 (B) 0.4
(C) 0.03 (D) none of the above

17. Which of the following statements is **true**?
(A) Baye's Theorem is also called the theorem of total probability
(B) Frequency and probability mean the same thing
(C) Mean $= \frac{1}{2}$ (3 median − mode)
(D) $P(B|A) = P(B)$ for any two events A and B

18. A husband and wife appear in an interview for two vacancies. The probability of hus band's selection is $\frac{1}{7}$ and that of wife's selection is selection is $\frac{1}{5}$. Then the probability that only one of them will be selected is
(A) $\frac{1}{35}$ (B) $\frac{34}{35}$
(C) $\frac{2}{7}$ (D) none of the above

19. Two independent events A and B are such that $P(A \cap B) = \frac{1}{6}$, $P(\overline{A} \cap \overline{B}) = \frac{1}{3}$ and $P(A) > P(B)$. Then $P(A) =$
(A) $\frac{1}{2}$ (B) $\frac{2}{3}$
(C) $\frac{1}{3}$ (D) $\frac{1}{4}$

20. The probability that it will rain during the rainy season of the year is 0.7. A person wants to decide whether or not to go out for a party tomorrow. The conditional probability that the rain is forecast given that it is the rainy season is 0.85. Then the probability that it will not rain tomorrow given that rain is forecast is
(A) 0.105 (B) 0.255
(C) 0.595 (D) none of these

21. A, B, C are three events in a sample space. ABC stands for $A \cap B \cap C$. Match List I (Description of Events) with List II (Expression) and choose the correct answer using the codes given below the Lists.

List I	List II
A. At least one of them does not occur	1. $\bar{A}BC \cup A\bar{B}C \cup AB\bar{C}$
B. Exactly two of them occur	2. $BC \cup CA \cup AB$
C. At least two of them occur	3. $\bar{A} \cup \bar{B} \cup \bar{C}$
D. None of them occurs	4. $\overline{(A \cup B \cup C)}$

Codes

	A	B	C	D
(A)	3	1	2	4
(B)	1	3	2	4
(C)	3	1	4	2
(D)	1	3	4	2

22. Five sticks of lengths 2, 4, 6, 8 and 10 feet are given. Three of these sticks are selected at random. The probability that the selected sticks can form a triangle is

(A) 0.5 **(B)** 0.4
(C) 0.3 **(D)** zero

23. Events $A_1, A_2, \ldots\ldots, A_n$ are independent and $P(A_i) = \dfrac{1}{1+i}$, for $1 \le i \le n$. Then the probability that none of these events occurs is

(A) $\dfrac{n}{n+1}$ **(B)** $\dfrac{1}{n+1}$
(C) $\dfrac{1}{(n+1)!}$ **(D)** $1 - \dfrac{1}{(n+1)!}$

24. The probabilities that A, B, C can solve a problem in programming are $\dfrac{4}{3}, \dfrac{3}{7}, \dfrac{2}{9}$ respectively. If all of them try then the probability that the problem is solved by only one of them is

(A) $\dfrac{4}{5}$ **(B)** $\dfrac{3}{7}$
(C) $\dfrac{2}{9}$ **(D)** $\dfrac{141}{315}$

25. 9 balls are drawn at random from a bag containing 11 white and 9 black balls. The probability that 5 are white is

(A) $\dfrac{{}^{11}C_5}{{}^{20}C_5}$ **(B)** $\dfrac{{}^{11}C_5 \times {}^{9}C_4}{{}^{20}C_5}$
(C) $\dfrac{{}^{11}C_5 + {}^{9}C_4}{{}^{20}C_5}$ **(D)** none of these

26. There are two bags, one of which contains 4 red and 6 green balls and the other 5 red and 4 green balls. A ball is to be drawn from one or other of the two bags. Then the chance of drawing a green ball is

(A) $\dfrac{3}{10}$ **(B)** $\dfrac{2}{9}$
(C) $\dfrac{47}{90}$ **(D)** none of these

27. There are three bags which are known to contain 3 white and 4 red, 2 white and 3 red and 4 white and 1 red balls respectively. A ball was drawn at random from one of the bags and found to be a white ball. Then the chance that it was drawn from the first bag is

(A) $\dfrac{5}{19}$ **(B)** $\dfrac{14}{57}$
(C) $\dfrac{28}{57}$ **(D)** none of these

28. In a batch of 15 students, 5 students failed in an examination. The marks of the 10 students who passed are
40, 50, 60, 70, 80, 70, 80, 90, 90, 70.
Then the median of all the students is

(A) 75 **(B)** 65
(C) 60 **(D)** 70

29. The measure of central tendency most affected by extreme values is

(A) Mode **(B)** G. M
(C) Median **(D)** Mean

30. The observations 1, 4, 5, 8, 7 have the standard deviation 2.45. Then the standard deviation of 10, 40, 50, 80, 70 will be

(A) 0.245 **(B)** 2.45
(C) 24.5 **(D)** $2.45 \times \sqrt{10}$

31. For an observed variable x,
$\Sigma x_i = 20$, $\Sigma x_i^2 = 200$ and $n = 10$.
Then the coefficient of variation (expressed as a percentage) equals
(A) 400 **(B)** 200
(C) 100 **(D)** 80

32. Classes A, B, C contain 30, 40 and 50 children respectively. The average ages of children in classes A and C are equal; the average age of children in class B is 6.6 years. If the average age of all the children in the three classes is 7.2 years then the average age of children in class C is
(A) 6.8 years **(B)** 7.0 years
(C) 7.5 years **(D)** 7.8 years

33. We are given two urns A, B. Urn A contains 3 red and 2 white marbles; B contains 2 red and 5 white balls. An urn is chosen at random, a marble is drawn and put into the other urn. Then a second marble is drawn from the second urn. The probability that both marbles drawn are of the same colour is
(A) $\frac{901}{1680}$ **(B)** $\frac{779}{1680}$
(C) $\frac{901}{779}$ **(D)** $\frac{779}{901}$

34. A person draws two balls from a bag containing 3 white and 5 maroon balls. If he is to receive 10 Rs for every white ball, he draws and 1 Re for every maroon ball, then his expectation is
(A) $\frac{15}{64}$ **(B)** $\frac{35}{8}$
(C) $\frac{53}{8}$ **(D)** none of these

35. Five flowers are drawn, one by one from a basket containing 6 red and 4 white flowers. Then the odds against the event of drawing 3 red and 2 white flowers equals
(A) $\frac{10}{21}$ **(B)** $\frac{11}{10}$
(C) $\frac{10}{11}$ **(D)** none of these

36. The odds against A solving a certain problem are 4 to 3 and the odds in favour of B solving the same problem are 7 to 5. Then the chance that the problem will be solved if they both try is
(A) $\frac{5}{21}$ **(B)** $\frac{4}{7}$
(C) $\frac{5}{12}$ **(D)** $\frac{16}{21}$

37. The chance of 3 marksmen of scoring a bull at a single shot are $\frac{1}{3}, \frac{1}{2}, \frac{1}{4}$ respectively. If all the three fire together, then the number of possible events is
(A) 1 **(B)** 3
(C) 6 **(D)** 8

38. A group of 12 persons take their seats at a round table. Then the odds against two specified persons sitting together are
(A) $\frac{2}{9}$ **(B)** $\frac{9}{2}$
(C) $\frac{2}{11}$ **(D)** none of these

39. A, B, C, D toss a coin in that order. The one who gets the head first wins. Then the probability of B winning is
(A) $\frac{8}{15}$ **(B)** $\frac{4}{15}$
(C) $\frac{2}{15}$ **(D)** $\frac{1}{15}$

40. A box contains a white and b black balls. c balls are drawn. Then the expected value of the number of white balls drawn is
(A) $\frac{a}{a+b}$ **(B)** $\frac{c}{a+b}$
(C) $\frac{ac}{a+b}$ **(D)** none of these

41. A box contains two coins one of which is fair and the other two headed. One coin is chosen at random and tossed twice. If two heads appear then the probability that the chosen coin was two headed is

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$

(C) $\frac{1}{3}$ (D) $\frac{4}{5}$

42. A and B are events with $P(A) = \frac{1}{4} = P(A|B)$ and $P(B|A) = \frac{1}{2}$. Which one of the following is **not** correct?
(A) A and B are mutually exclusive
(B) A and B are independent
(C) $P(\overline{A}|B) = \frac{3}{4}$
(D) $P(\overline{B}|A) = \frac{1}{2}$

43. A and B are any two events in a sample space S. If $P(A \cap B) \geq P(A) + P(B) + \alpha$, then α is
(A) -1 (B) 1
(C) 0 (D) 2

44. Three houses are advertised to be let out in a certain locality and three persons made separate applications for a house. Then the probability that all the three persons made applications for the same house is

(A) $\frac{1}{9}$ (B) $\frac{2}{9}$

(C) $\frac{1}{3}$ (D) $\frac{2}{3}$

45. A box contains 2^n tickets among which ${}^nC_i$ tickets bear the number i, i = 0, 1,..., n. A set of n tickets is drawn. Then the expectation of the sum of their numbers is
(A) $\frac{mn}{2}$ (B) mn
(C) m + n (D) none of these

46. Out of 20 consecutive numbers two numbers are chosen at random. Then the probability that their sum is odd equals

(A) $\frac{9}{19}$ (B) $\frac{10}{19}$

(C) $\frac{9}{10}$ (D) none of these

47. Which of the following statements is false?
(A) If arithmetic mean = 25 and harmonic mean = 9, then geometric mean = 15 for any two positive values.
(B) If A and B are independent events, then $\overline{A}$ and $\overline{B}$ are also independent.
(C) A negative coefficient of skewness implies that the mean is greater than the mode.
(D) If A and B are any two events, then $P(A \cup B) \leq P(A) + P(B)$

48. Thirteen cards are drawn simultaneously from a deck of 52 cards. If aces count 1, face cards 10 and others count by their denominations then the expectation of the total score on 13 cards is

(A) $\frac{1}{13}$ (B) $\frac{45}{13}$

(C) $\frac{40}{13}$ (D) $\frac{85}{13}$

49. In a multiple-choice test a candidate either knows the correct answer with probability p or guesses with probability $1 - p$. The probability of answering a question correctly is one if he knows the answer and $\frac{1}{m}$, if he merely guesses. If a candidate answers a question correctly then the probability that he really knows the answer is

(A) $\frac{mp}{1 + mp}$ (B) $\frac{mp}{1 + (m-1)p}$

(C) $\frac{(m-1)p}{1+(m-1)p}$ (D) $\frac{(m-1)p}{1+mp}$

50. Given the information that in a family with two children that at least one of the two is a boy, the probability that both are boys is

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$

(C) $\frac{1}{3}$ (D) none of these

51. From integers 1 to 40 (both inclusive) one integer is chosen at random. Then the probability that it is divisible by either 5 or 4 is

(A) $\frac{2}{5}$ (B) $\frac{1}{5}$

(C) $\frac{3}{5}$ (D) $\frac{4}{5}$

KEY

1. (A)	2. (B)	3. (D)	4. (D)
5. (D)	6. (D)	7. (D)	8. (A)
9. (B)	10. (C)	11. (C)	12. (B)
13. (B)	14. (C)	15. (C)	16. (B)
17. (C)	18. (C)	19. (A)	20. (B)
21. (A)	22. (C)	23. (B)	24. (D)
25. (B)	26. (C)	27. (A)	28. (C)
29. (D)	30. (C)	31. (B)	32. (C)
33. (A)	34. (B)	35. (B)	36. (D)
37. (D)	38. (B)	39. (B)	40. (C)
41. (D)	42. (A)	43. (A)	44. (A)
45. (A)	46. (B)	47. (C)	48. (D)
49. (B)	50. (C)	51. (A)	

EXPLANATORY NOTES

19. A, B are independent

$\Rightarrow \overline{A}, \overline{B}$ are independent

$$\Rightarrow P(\overline{A} \cap \overline{B}) = P(\overline{A}) \cdot P(\overline{B})$$

$$\Rightarrow \frac{1}{3} = [1 - P(A)]\,[1 - P(B)]$$

$$\Rightarrow \frac{1}{3} = 1 - P(A) - P(B) + P(A) \cdot P(B)$$

$$\Rightarrow -\frac{2}{3} = -P(A) - P(B) + \frac{1}{6}$$

$[\because$ A, B are independent,

$$\frac{1}{6} = P(A \cap B) = P(A) \cdot P(B)]$$

$$\Rightarrow P(A) + P(B) = \frac{5}{6}$$

$$\Rightarrow P(B) = \frac{5}{6} - x \text{ where } x = P(A)$$

Since $P(A) \cdot P(B) = \frac{1}{6}$, $x\left(\frac{5}{6} - x\right) = \frac{1}{6}$

or $6x^2 - 5x + 1 = 0$

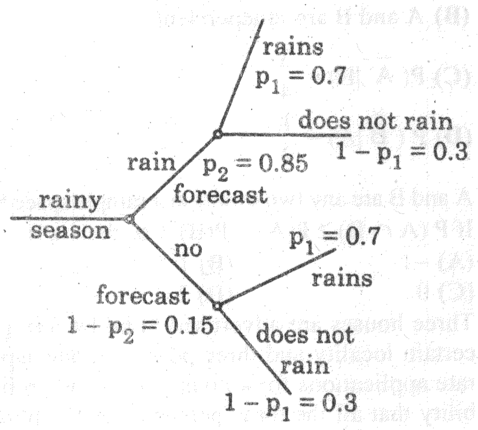
i.e., $(2x - 1)(3x - 1) = 0$

$\therefore x = \frac{1}{2}, \frac{1}{3}$

When $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$. Also $P(A) > P(B)$ is satisfied.

Note that when $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$. $P(A) > P(B)$ is not satisfied.

20. See tree diagram given below,



given $p_1p_2 + p_1(1 - p_2) = 0.7$

or $p_1 = 0.7$

$p_2 = 0.85$ (given)

Required probability

$= P(\text{not rains}|\text{rain is forecast})$

$= 0.3 \times 0.85 = 0.255$

(Events of raining and forecast are independent events i.e., $p_1 = 0.7$).

22. For a triangle the sum of any two sides must be greater than the third. Hence the possible combinations for sides are (4, 6, 8), (6, 8, 10), (4, 8, 10) out of ${}^5C_3$ = 10 ways of choosing three sticks which are (2, 4, 6), (2, 4, 8), (2, 4, 10), (2, 6, 8), (2, 6, 10) (2, 8, 10), (4, 6, 8), (4, 6, 10), (6, 8, 10) and (4, 8, 10).

$\therefore$ required probability $= \frac{3}{10} = 0.3$

23. $A_1, A_2, \ldots\ldots, A_n$ are independent

$\Rightarrow \overline{A_1}, \overline{A_2}, \ldots$ are also independent

$$\Rightarrow P(\bar{A}_1 \bar{A}_2 \ldots \bar{A}_n) = P(\bar{A}_1)\, P(\bar{A}_2) \ldots P(\bar{A}_n)$$

$$= \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\ldots\left(1-\frac{1}{1+n}\right)$$

$$\left[\because P(A_i) = \frac{1}{1+i} \Rightarrow P(\bar{A}_i) = \left(1-\frac{1}{1+i}\right)\right]$$

$$= \frac{1}{2}\times\frac{2}{3}\times\ldots\times\frac{n}{n+1}$$

$$= \frac{1}{n+1}$$

24. Required probability
= Probability that A solves it and B, C do not
+ Probability that B solves it and C, A do not
+ Probability that C solves it and A, B do not

$$= \frac{4}{5}\times\frac{4}{7}\times\frac{7}{9}+\frac{1}{5}\times\frac{3}{7}\times\frac{7}{9}+\frac{1}{5}\times\frac{4}{7}\times\frac{2}{9}$$

$$= \frac{112+21+8}{315} = \frac{141}{315}$$

25. 9 balls can be drawn from 20 balls in ${}^{20}C_5$ ways.5 white balls can be drawn from 11 white balls in ${}^{11}C_5$ ways and 4 black balls from 9 black balls in ${}^{9}C_4$ ways. So, the number of favourable cases is ${}^{11}C_5 \times {}^{9}C_4$.

$$\therefore \text{ required probability} = \frac{{}^{11}C_5 \times {}^{9}C_4}{{}^{20}C_5}$$

26. The chance that the first bag is chosen is $\frac{1}{2}$ and the chance of drawing a green ball from it is $\frac{6}{10}$.

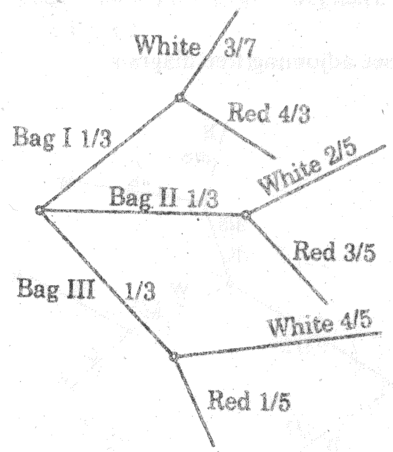
$\therefore$ The chance of choosing the first bag and drawing a green ball is $\frac{1}{2}\times\frac{6}{10}$.

Similarly the chance of choosing the second bag and drawing a green ball is $\frac{1}{2}\times\frac{4}{9}$.

$\therefore$ required probability

$$= \frac{1}{2}\times\frac{6}{10}+\frac{1}{2}\times\frac{4}{9} = \frac{47}{90}$$

27. See adjoining tree diagram



Required probability

$$= \frac{\frac{1}{3}\times\frac{3}{7}}{\frac{1}{3}\times\frac{3}{7}+\frac{1}{3}\times\frac{2}{5}+\frac{1}{3}\times\frac{4}{5}}$$

$$= \frac{5}{19}$$

28. Denote by m_1, m_2, m_3, m_4, m_5 the marks of the 5 failed students in ascending order.

$$\underbrace{m_1, m_2, m_3, m_4, m_5, 40, 50,}\ 60, \underbrace{70, 70, 70, 80, 80, 90, 90}$$

$\downarrow$

median

$\therefore$ median = 60

30. $\sigma_1^2 = 10^2 \times \sigma^2 = 10^2 \times (2.45)^2$

$\therefore \sigma_1 = 10 \times 2.45 = 24.5$

31. $\bar{x} = \frac{\Sigma x_i}{n} = 2$

$$\sigma^2 = \frac{\Sigma x_i^2}{n} - \bar{x}^2 = \frac{200}{10} - 4 = 16$$

$\therefore \sigma = 4$

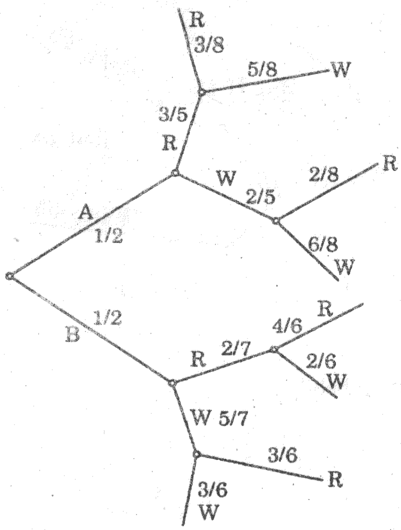
$$\text{C. V.} = \frac{\sigma}{\bar{x}} \times 100 = 200$$

32. Let x= average age of children in class C

Then $(30 + 50)\,x + 6.6 \times 40 = 120 \times 7.2$

$\therefore x = 7.5$

33. See adjoining tree diagram.



Required probability

$$= \frac{3}{8}\times\frac{3}{5}\times\frac{1}{2} + \frac{6}{8}\times\frac{2}{5}\times\frac{1}{2} + \frac{4}{6}\times\frac{2}{7}\times\frac{1}{2} + \frac{3}{6}\times\frac{5}{7}\times\frac{1}{2}$$

$$= \frac{901}{1680}$$

34. X = random variable showing amount.

$= 10, 1$

$E(x) = 10 \times$ probability of white $+ 1 \times$ probability of maroon

$$= 10 \times \frac{3}{8} + 1 \times \frac{5}{8} = \frac{35}{8}$$

35. p = Probability of drawing 3 red and 4 white flowers

$$= \frac{{}^6C_3 \times {}^4C_2}{{}^{10}C_5} = \frac{10}{21}$$

$$\therefore q = 1 - p = \frac{11}{21}$$

Odds against the event $= \dfrac{q}{p} = \dfrac{11}{10}$

36. For A, $\dfrac{q}{p} = \dfrac{4}{3}$ where $q = 1 - p$

$\therefore p = \dfrac{3}{7}$ = probability of A solving the problem.

For B, $\dfrac{p}{q} = \dfrac{7}{5}$ where $q = 1 - p$

$\therefore p = \dfrac{7}{12}$ = probability of B solving the problem.

Now probability that neither A nor B solves the problem $= \left(1 - \dfrac{3}{7}\right)\times\left(1 - \dfrac{7}{12}\right)$

$$= \frac{4}{7}\times\frac{5}{12} = \frac{5}{21}$$

$\therefore$ probability that the problem will be solved (either by A or by B or by both)

$$= 1 - \frac{5}{21} = \frac{16}{21}$$

37. The possible events are

(i) all the three score	= 1 event
(ii) only two of them score	= 3 events
(iii) only one of them scores	= 3 events
(iv) none scores	= 1 event
	Total = 8 events

38. Let A and B be the two specified persons. A having taken his place, B has a choice of 11 seats, 2 of which are next to A (to the right and left of A). So the odds against B sitting next to A are $\dfrac{q}{p} = \dfrac{9/11}{2/11} = \dfrac{9}{2}$

39. p = probability of getting a head $= \dfrac{1}{2}$

$\therefore q = 1 - p = \dfrac{1}{2}$.

B gets his chance in the 2nd, 6th, 10th, ... tosses.

$\therefore$ probability that B wins

$= qp + q^5p + q^9p + \ldots\ldots$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^{10} + \ldots\ldots$$

$$= \frac{a}{1-r} \text{ where } a = \frac{1}{2^2}, r = \frac{1}{2^4}$$

(geometric series formula)

$$= \frac{\frac{1}{2^2}}{1 - \frac{1}{2^4}} = \frac{4}{15}$$

40. Define a random variable x_i as follows:-

$$x_i = \begin{cases} 1 \text{ if ith ball drawn is white} \\ 0 \text{ if ith ball drawn is black.} \end{cases}$$

Let S = number of white balls among the c balls drawn

$$= \sum_{i=1}^{c} x_i$$

Then $E(S) = \sum_{i=1}^{c} E(x_i)$

Now $E(x_i) = 1 \times P(x_i = 1) + 0 \times P(x_i = 0)$

$$= \frac{a}{a+b}$$

$$\therefore E(S) = \sum_{i=1}^{c} E(x_i) = \sum_{i=1}^{c} \frac{a}{a+b} = \frac{ac}{a+b}$$

41. Fair coin H T; other coin H H

Probability of choosing the fair coins and getting 2 heads $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

Probability of choosing the other coins and getting 2 heads $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

($\because$ both faces of the coin are H, H and appearance of a head is a sure event)

$\therefore$ probability of 2 heads $= \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$

$$\text{Required Probability} = \frac{\frac{1}{2}}{\frac{5}{8}} = \frac{4}{5}$$

42. $\frac{1}{4} = P(A|B) = \frac{P(A \cap B)}{P(B)}$ and ...(1)

$\frac{1}{2} = P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{1/4}$ ...(2)

From (2), $P(A \cap B) = \frac{1}{8} \neq 0$

$\therefore$ A, B are not mutually exclusive.

From (1), $P(B) = \frac{1}{8} \times 4 = \frac{1}{2}$

Thus $P(A) \cdot P(B) = P(A \cap B)$ i.e., A, B are independent.

Again $P(\overline{A}|B)$

$$= \frac{P(\overline{A} \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)}$$

$$= \frac{\frac{1}{2} - \frac{1}{8}}{1/2} = \frac{3}{4}$$

and $P(\overline{B}|A)$

$$= \frac{P(A) - P(A \cap B)}{P(A)} = \frac{\frac{1}{4} - \frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$$

43. $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$

$\therefore P(A) + P(B) - 1 \leq P(A \cap B)$

or $P(A \cap B) \geq P(A) + P(B) - 1$

$\therefore \alpha = -1$

44. First person can apply for any of the three houses, second can also apply to any of the three houses and the third one can also apply to any of the three houses. So, total number of ways of applying = 3 × 3 × 3 = 27. Since there are 3 houses all the three persons can apply to the same house in 3 different ways.

$\therefore$ required probability $= \frac{3}{27} = \frac{1}{9}$

45. Let x_i, i = 1, 2, ... m, be a random variable representing the number on the ith ticket drawn.

Let $S = x_1 + x_2 + ... + x_m$.

Then $E(S) = \sum_{i=1}^{m} E(x_i)$

Now x_i can have anyone of the values 0, 1, 2, ... n.

There are ${}^nC_0$ tickets having the number 0.

$\therefore P(x_i = 0) = \frac{{}^nC_0}{2^n}$.

Similarly $P(x_i = 1) = \frac{{}^nC_1}{2^n}$ ......

$P(x_i = n) = \frac{{}^nC_n}{2^n}$

$\therefore E(x_i) = 0 \times P(x_i = 0) + 1 \times P(x_i = 1) + ... + n \times P(x_i = n)$

$$= \frac{{}^nC_1}{2^n} + \frac{2 \times {}^nC_2}{2^n} + ... + n \times \frac{{}^nC_n}{2^n}$$

$$= \frac{1}{2^n}\left[{}^nC_1 + 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 + ... + n \cdot {}^nC_n\right]$$

$$= \frac{n}{2^n}\left[1 + \frac{n-1}{1!} + \frac{(n-1)(n-2)}{2!} + ... + 1\right]$$

$$= \frac{n}{2^n}\left[{}^{n-1}C_0 + {}^{n-1}C_1 + ... + {}^{n-1}C_{n-1}\right]$$

$$= \frac{n}{2^n}[1+1]^{n-1} \text{ (by Binomial Theorem)}$$

$$= \frac{n}{2}$$

Thus $E(S) = \sum_{i=1}^{m} E(x_i) = \sum_{i=1}^{m} \frac{n}{2} = \frac{mn}{2}$

46. 2 numbers can be chosen from 20 numbers in ${}^{20}C_2$ ways. Sum of the numbers is odd when one of them is odd and the other even. There are 10 odd and 10 even numbers. One odd number can be chosen in 10 ways and one even number chosen in 10 ways.

$\therefore$ required probability $= \frac{10 \times 10}{{}^{20}C_2} = \frac{10}{19}$

47. $(G.M)^2 = (A.M) \times (H.M) = 25 \times 9 = 225$

$\therefore G.M = 15$

A, B independent $\Rightarrow \bar{A}, \bar{B}$ independent.

A negative coefficient of skewness indicates that the distribution is negatively skewed. i.e., it has a longer tail towards the left and so mean < median < mode. For any two events A, B, $P(A \cap B) \geq 0$ and

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$\leq P(A) + P(B)$.

48. Let x_i stands for the number corresponding to the ith card. Face cards are given, king and jack cards. Then x_i takes the values 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10. Probability for each card of the same type $= \frac{4}{52} = \frac{1}{13}$

Let $S = \sum_{i=1}^{13} x_i$. Then $E(S) = \sum_{i=1}^{13} E(x_i)$

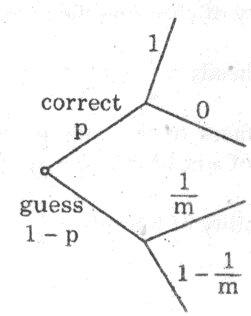
Now $E(x_i) = \frac{1}{13} \times 1 + \frac{1}{13} \times 2 + ...$

$$+ \frac{1}{13} \times 10 + \frac{1}{13} \times 10 + \frac{1}{13} \times 10 + \frac{1}{13} \times 10$$

$$= \frac{1}{13}(1 + 2 + ... + 9 + 40) = \frac{85}{13}$$

49. The candidate can answer the question correctly in two mutually exclusive ways, namely, (i) knowing the correct answer (ii) guessing the correct answer.

Required probability (see tree diagram)



$$= \frac{p \cdot 1}{p \cdot 1 + (1-p)\frac{1}{m}} = \frac{mp}{1+(m-1)p}$$

50. Sample space = {BB, BG, GB, GG}
Reduced sample space = {BB, BG, GB}.
Thus there are 3 ways in which at least one is a boy.
Both are boys happen in only one way.

$\therefore$ required probability $= \frac{1}{3}$

51. A = {5, 10, ... 40} and B = {4, 8, ... 40}

$A \cap B = \{20, 40\}$. So $P(A) = \frac{8}{40}$, $P(B) = \frac{10}{40}$,

$P(A \cap B) = \frac{2}{40}$

$\therefore$ required probability
$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{2}{5}$

2.4 PROBABILITY DISTRIBUTIONS

(A) Discrete Distributions

1. The Binomial Distributions

Trials of an experiment are said to be **independent** if the outcome of any trial does not depend upon the previous outcomes.

Consider an experiment with first two outcomes. We call one of the outcomes success and the other outcome failure. Independent repeated trials of such an experiment are known as Bernouilli trials, named after the discoverer Jacob Bernouilli. For example, consider the experiment of tossing a fair coin. If appearance of a head is termed as success, then appearance of tail will be failure.

Let p = Probability of success in a Bernouilli trial. Then q = 1 – p = probability of failure.

A **Binomial experiment** denoted by B(n, p), consists of a fixed number of trials n and probability of success p.

The probability of exactly r successes in a Binomial experiment B (n, p) is given by

$$P(r) = P(r \text{ success}) = {}^nC_r\, p^r q^{n-r}$$

where $q = 1 - p$.

Definition 1: A discrete random variable X is said to have a **binomial** or a **Bernouilli distribution** if its probability density function is given by

$P(X = r) = {}^nC_r\, p^r q^{n-r}$, $r = 0, 1, 2, \ldots. n$ where $q = 1 - p$, $0 < p < 1$.

Remarks: 1. The Binomial distribution is known completely when n and p are given. Thus there are two parameters n and p for the distribution.

2. For the Binomial distribution B (n, p).

(i) Expected value E(x) or Mean = np

(ii) Variance V(x) = npq

(iii) Third (central) moment = npq (q – p)

3. Additive property of Binomial Distributions

If x_1, x_2 are independent binomial variables with parameters (n_1, p) and (n_2, p) respectively, then $x_1 + x_2$ is also a binomial variable with parameters $(n_1 + n_2, p)$.

Example: A man fires at a target 6 times; the probability of his hitting it is equal to 0.4.

(i) What is the probability that he will hit the target at least once?

(ii) How many times must he fire at the target so that the probability of hitting is at least once is greater than 0.77?

Solution

Here p = 0.4, q = 0.6 and n = 6

$\therefore P(X = x) = {}^nC_x\, p^x q^{n-x} = {}^6C_x (0.4)^x (0.6)^{6-x}$

(i) Probability of hitting at least once
$= P(x \geq 1) = 1 - P(x = 0) = 1 - (0.6)^6$

(ii) Suppose n is the required number of times.

Given $1 - (0.6)^n > 0.77$ or $0.23 > (0.6)^n$.
This is satisfied for $n \geq 3$. So the man must fire at the target three times or more.

Example: A Student takes an objective type question examination with four choices for each questions. There are 72 questions. Assume that one of the choices is obviously wrong. The student makes a guess of the remaining choices with his knowledge of the subject matter. Find the expected number and variance of success.

Solution

Here n = 72, $p = \frac{1}{3}$, $q = \frac{2}{3}$,
so that expectation = np = 24 and variance = npq = 16.

2. The Poisson Distribution

Poisson distribution is the limiting form of a Binomial distribution. It is the discovery of the French Mathematician Poisson. As n becomes very large ($n \to \infty$) and p becomes very small ($p \to 0$) in such a way that np is a finite quantity λ, the Poisson distributions results.

Definition 2: A discrete random variable X is said to follow **Poisson distribution** if

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2 \ldots\ldots \infty$$

λ is known as the parameter of the distribution.

Poisson distribution is described as a distribution of rare events. Examples are

(a) number of deaths in a battallion of soldiers due to the kick of a horse.

(b) number of faulty blades in a pack of 1000 blades.

(c) number of cars passing a crossing per minute during the busy hours of a day.

Remarks: 1. For a Poisson distribution with parameter λ, mean = variance.

2. (Additive or reproductive property of Poisson distribution)

If x_1, x_2 are independent Poisson variables with parameters λ_1, λ_2 respectively, then $x_1 + x_2$ is also a Poisson variable with parameter $\lambda_1 + \lambda_2$. That is, the Poisson distribution reproduces itself by addition.

Example: If a random variable X has a Poisson distribution such that

$P(x = 1) = P(x = 2)$, find $P(x = 4)$.

Solution

$$P(x = 1) = P(x = 2) \Rightarrow \frac{e^{-\lambda}\lambda}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$$

$$\Rightarrow \lambda = 2 \quad (\because \lambda \neq 0)$$

$$\therefore P(x = 4) = \frac{e^{-2}2^4}{4!} = \frac{2}{3}e^{-2}$$

Example: A certain Electronics Company produces a particular type of vacuum tube. It has been observed that on the average, 3 tubes out of 100 are defective. The company packs the tubes in boxes of 400. What is the probability that a box of 400 tubes will contain (a) at least k defectives (b) at most one defective?

Solution

Here n = 400, $P = \frac{3}{100}$ $\quad \therefore \lambda = np = 12$

(a) P(at least k defectives) = $P(x \geq k)$

$= P(x = k) + P(x = k + 1) + \ldots + P(x = 400)$

$$= \sum_{x=k}^{400} \frac{e^{-12}12^x}{x!}$$

(b) P(at most one defective) = $P(x = 0) + P(x = 1)$

$= e^{-12} + e^{-12} \times 12$

$= 13e^{-12}$.

Example: A random variable X has a Poisson distribution with $P(X = 2) = \frac{2}{3}P(X = 1)$. Find $P(X = 0)$.

Solution

We know that $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2\ldots$

$$\therefore P(X = 2) = \frac{2}{3}\ P(X = 1)$$

$$\Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = \frac{2}{3} \times \frac{e^{-\lambda}\lambda}{1!}$$

$$\Rightarrow \lambda = \frac{4}{3}$$

Thus $P(X = 0) = e^{-\lambda} = e^{-4/3}$

Example: If X has a Poisson distribution with $E(X^2) = 2$ then $P(X = 0)$ equals

(a) $P(X = 4)$ **(b)** $P(X = 3)$

(c) $P(X = 2)$ **(d)** $P(X = 1)$

Solution

Since X follows Poisson distribution,

Variance X = Mean X

i.e., $E(X^2) - [E(X)]^2 = E(X)$

Put $E(X) = \lambda$.

Then $2 - \lambda^2 = \lambda$ $\quad (\because E(X^2) = 2)$

or $\lambda^2 + \lambda - 2 = 0$

$\because (\lambda + 2)(\lambda - 1) = 0$

$\therefore \lambda = 1, -2$. $\lambda = -2$ is impossible ($\because$ Variance is +ve)

So $P(X = 0) = e^{-\lambda} = e^{-1}$ and

$$P(X = 1) = \frac{e^{-1}1^1}{1!} = e^{-1}$$

$\therefore P(X = 0) = P(X = 1)$

So correct answer is (d).

Example: The probability of an item being defective is 0.01. What is the probability that a sample of 100 items randomly selected will contain not more than one defective item?
(Assume $e^{-1} = 0.36788$)

Solution

Here $P = 0.01, n = 100$

$\therefore \lambda = np = 1$

Let X be the Poisson random variable representing the number of defective items.

Then, $P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{x!} = \dfrac{e^{-1}}{x!}$

$\therefore$ probability is not more than one defective item

$$= P(x \le 1) = P(x = 0) + P(x = 1)$$

$$= \frac{e^{-1}}{0!} + \frac{e^{-1}}{1!} = 2e^{-1} \; (\because 0! = 1 \text{ and } 1! = 1)$$

$$= 2 \times 0.36788 = 0.73576$$

Example: A random variable X follows Poisson Distribution with parameter 4. Then the probability that X assumes the values less than 3 is (Given that $e^{-4} = 0.0183$)

(a) 0.7621 (b) 0.1952
(c) 0.0915 (d) none of these

Solution

$$P(X = x) = \frac{e^{-4}4^x}{x!}, x = 0, 1, 2 \ldots$$

$$\therefore P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$$

TABLE OF VALUES OF e^{-m}
(m lying between 0 and +1)

m	0	1	2	3	4	5	6	7	8	9
0.0	1.000	.9900	.9802	.9704	.9608	.9512	.9418	.9324	.9231	.9139
0.1	.9048	.8958	.8869	.8781	.8694	.8607	.8521	.8437	.8353	.8207
0.2	.8187	.8106	.8025	.7945	.7866	.7788	.7711	.7634	.7558	.7483
0.3	.7408	.7334	.7261	.7189	.7118	.7047	.6977	.6907	.6839	.6771
0.4	.6703	.6636	.5570	.6505	.6440	.6376	.6313	.6250	.6188	.6126
0.5	.6065	.6005	.5945	.5886	.5827	.5770	.5712	.5655	.5599	.5543
0.6	.5488	.5434	.5379	.5326	.5273	.5220	.5169	.5117	.5066	.5016
0.7	.4966	.4916	.4868	.4819	.4771	.4724	.4677	.4630	.4584	.4538
0.8	.4493	.4449	.4404	.4360	.4317	.4274	.4232	.4190	.4148	.4107
0.9	.4066	.4025	.3985	.3946	.3906	.3867	.3829	.3791	.3753	.3716

For example $e^{-.5} = 0.6065$. The same value was obtained in the previous example. Similarly, $e^{-.57} = 0.5655$.

TABLE OF e^{-m}
(m lying between +1 and +10)

e- m	1	2	3	4	5
	0.36788	0.13534	0.04979	0.01832	0.006738
	6	7	8	9	10
	0.002479	0.000912	0.000335	0.000123	0.00045
Example $e^{-1.5} = e^{-1} \times e^{-.5} = .36788 \times .6065 = .22311922$ or $.2231$					

$$= e^{-4} + \frac{e^{-4} \times 4}{1!} + \frac{e^{-4} \times 4^2}{2!}$$

$$= e^{-4}(1+4+8)$$

$$= 0.0183 \times 13 = 0.2379$$

So correct answer is (d)

Example: If 5% of the electric bulbs manufactured by a company are defective, find the probability that in a sample of 100 bulbs (i) none is defective (ii) at least 3 are defective (given $e^{-5} = 0.006738$)

Solution

$$p = \frac{5}{100} = 0.05; n = 100 \therefore np = 5 = \lambda$$

$$\therefore P(X = x) = \frac{e^{-5} 5^x}{x!}$$

(i) Probability that none is defective

$$= P(X = 0) = e^{-5} = 0.006738$$

(ii) Probability that at least 3 are defective

$$= P(X \geq 3) = 1 - P(X < 3)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - e^{-5}\left(1 + 5 + \frac{5^2}{2}\right)$$

$$= 1 - 0.006738 \times 18.5$$

$$= 0.875347$$

(B) Continuous Distribution

Let X be a continuous random variable. If f(x) is the probability density functions of X, then

(i) $P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$ ...(1)

(ii) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx$ ...(2)

(iii) Expectation of X = E(X)

$$= \text{Mean} = \int_{-\infty}^{\infty} x f(x)\,dx \quad ...(3)$$

(iv) Variance of X = y (X)

$$= \int_{-\infty}^{\infty} [x - E(x)^2 f(x)dx] \quad ...(4)$$

(v) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

with the above ideas we discuss some continuous distributions.

3. The Uniform Distribution

Definition 3: A continuous random variable X is said to follow **uniform distribution** or **rectangular distribution** if its probability density function is given by,

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

Remarks: 1. The uniform distribution generally arises in the study of round off errors where measurements are recorded upto a certain level of accuracy.

2. The cumulative distribution function of X is given by $f(x) = p(X \leq x)$.

$$= \int_{-\infty}^{x} f(x)dx$$

$$= \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$$

3. Mean $= \frac{a+b}{2}$, Variance $= \frac{(b-a)^2}{12}$

Example: The melting point X of a certain specimen may be assumed to be a continuous random variable which is **uniformly** distributed over the interval [110, 120]. Find the density function of X, mean of X, variance of X and $P(112 \leq x \leq 115)$.

Solution

Here a = 110, b = 120 so that density function is

$$f(x) = \begin{cases} \frac{1}{120-110}, & 110 \leq x \leq 120 \\ 0; & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{10}, & 110 \le x \le 120 \\ 0, & \text{otherwise} \end{cases}$$

Mean of $X = \frac{a+b}{2} = \frac{110+120}{2} = 115$

Variance of X

$$= \frac{(b-a)^2}{12} = \frac{(120-110)^2}{12} = \frac{25}{3}$$

$P(112 \le x \le 115)$

$$= \int_{112}^{115} f(x)\,dx + \int_{112}^{115} \frac{1}{10}\,dx = \frac{3}{10}$$

Example: The efficiency X of a certain electrical component may be assumed to be a random variable which is **uniformly** distributed between 0 and 100 units.

Find the probability that X is

(i) between 60 and 80 units and

(ii) greater than 90 units.

Solution

$$\text{Here } f(x) = \begin{cases} \frac{1}{100}, & 0 \le x \le 100 \\ 0, & \text{elsewhere} \end{cases}$$

$$\therefore P(60 \le X \le 80) = \int_{60}^{80} f(x)\,dx$$

$$= \frac{1}{100}\int_{60}^{80} dx = \frac{1}{5}$$

$$\text{and } P(x > 90) = \int_{90}^{100} f(x)\,dx$$

$$= \frac{1}{100}\int_{90}^{100} dx = \frac{1}{10},$$

Thus 20% of the electrical components will have an efficiency between 60 and 80 units and 10% will have an efficiency which is more than 90 units.

Example: If X is uniformly distributed in [– 2, 2] then P (X < 1) is

(a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 0

Solution

$$\text{Given } f(x) = \begin{cases} \frac{1}{4}, & -2 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$\therefore P(X<1) = \int_{-2}^{1} f(x)\,dx = \frac{1}{4}\,(x)_{-2}^{1} = \frac{3}{4}$$

So the correct answer is (a).

Example: Trains arrive at station A at 15 minutes intervals starting at 5 A.M. If a passenger arrives at the station at a time that is uniformly distributed between 9 A.M and 9.30 A.M. Find the probability that he has to wait for the train for (a) less than 6 minutes (b) more than 10 minutes.

Solution

Let X stand for the random variable representing the number of minutes after 9 A.M that the passenger arrives at station A.

$$\therefore f(x) = \frac{1}{30}$$

(a) The passenger has to wait for less than 6 minutes

If he arrives at the station between 9.09 and 9.15 or between 9.24 and 9.30.

Probability for this = P (9 < x < 15) + P (24 < X < 30)

$$= \int_{9}^{15} \frac{1}{30}\,dx + \int_{24}^{30} \frac{1}{30}\,dx = \frac{2}{5}$$

(b) Probability in this case = P(0 < x < 5) + P (15 < x < 20)

$$= \int_{0}^{5} \frac{1}{30}\,dx + \int_{15}^{20} \frac{1}{30}\,dx = \frac{1}{3}$$

(4) The Exponential Distribution

Definition 4: A continuous random variable X is said to have the **exponential distribution** if its probability density function is given by

$$f(x)=\begin{cases} ae^{-ax}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

a being some positive number, called its parameter.

Remarks: 1. This distribution arises in the study of life-length of electronic components. Also the inter-arrival time and service time in queuing theory have exponential distributions.

2. Mean = $\frac{1}{a}$, Variance = $\frac{1}{a^2}$

Example: The sales tax return of a salesman is **exponentially** distributed with parameter $\frac{1}{4}$.

What is the probability that his sale will exceed Rs. 10,000, assuming that sales tax is charged at the rate of 5% on the sales?

Solution

Here $f(x)=\begin{cases} \frac{1}{4}e^{-\frac{x}{4}}, & x>0 \\ 0, & \text{otherwise} \end{cases}$

Sales tax for the sale of Rs. 10,000 is = $10000\times\frac{5}{100}$ = Rs. 500

Now $P(x>500) = 1-P(x\le 500)$

$$=1-\int_0^{500} f(x)\,dx = 1-\frac{1}{4}\int_0^{500} e^{\frac{-x}{4}}\,dx$$

$$=1+=\left(e^{\frac{-x}{4}}\right)_0^{500} e^{-125}$$

Example: At a construction site, 3 lorries unload materials per hour on an average. Find the probability that the time between arrival of successive lorries is (a) at least 30 minutes (b) less than 20 minutes.

Solution

Let X denote the exponential variable time in hours.

$$f(x)=\begin{cases} 3e^{-3x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

(a) $P(X\ge 30 \text{ minutes}) = P(X\ge \frac{1}{2} \text{ an hour})$

$$=\int_{\frac{1}{2}}^{\infty} 3e^{-3x}\,dx = e^{-1.5} = 0.22$$

(b) $P(x<\frac{1}{3}) = \int_0^{1/3} 3e^{-3x}\,dx = 1-e^{-1}$

$= 1-0.37 = 0.63$

Example: The density function of the time that Mr. X speaks over phone is given by

$$f(x)=\begin{cases} ke^{-x/6}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that Mr. X will take for
(a) more than 8 minutes (b) less than 6 minutes
(c) between 6 and 8 minutes.

Solution

Since f(x) is a density function

$$\int_0^{\infty} f(x)\,dx = 1$$

$$\therefore \int_0^{\infty} ke^{-x/6}\,dx = 1$$

$$\therefore -6k\left(e^{-x/6}\right)_0^{\infty} = 1$$

$$-6k(0-1)=1$$

$$\therefore k=\frac{1}{6}$$

Thus $f(x)=\begin{cases} \frac{1}{6}e^{-x/6}, & x>0 \\ 0, & \text{otherwise} \end{cases}$

Example: If X is an exponentially distributed random variable with parameter θ. Then the probability that X exceeds its mean value is

(a) 0.3678 **(b)** 0.2245
(c) 0.6322 **(d)** none of these

Solution

$$\text{Given } f(x) = \begin{cases} \theta e^{-\theta x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Mean} = \frac{1}{\theta}$$

$$\therefore P(X > \frac{1}{\theta}) = \int_{\frac{1}{\theta}}^{\infty} \theta e^{-\theta x} dx$$

$$= -\left(e^{-\theta x}\right)_{\frac{1}{\theta}}^{\infty} = e^{-1} = 0.3678$$

So the correct answer is (a).

(5) The Normal Distribution

Definition 5: A continuous random variable X is said to follow **normal distribution** if its probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad ; -\infty \le x \le \infty,$$

$$\sigma > 0, -\infty < \mu < \infty \qquad ...(1)$$

We write in such a situation $X \sim N(\mu, \sigma^2)$.

The distribution involves two parameters μ and σ.

Properties

1. The distribution is symmetrical.
2. Mean $= \mu$, Variance $= \sigma^2$
3. For this distribution, mean, median and mode coincide.
4. $f(x) \ge 0$ for all x
5. $\int_{-\infty}^{\infty} f(x)\, dx = 1$, i.e., the total area under, the curve $y = f(x)$ bounded by the axis of x is 1.

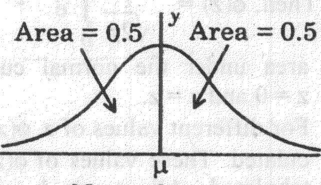
Remarks

1. The curve $y = f(x)$, known as the **normal curve**, is a bell-shaped curve. It is symmetrical about $x = \mu$; the two tails on the left and right sides of the mean extend to infinity.

2. Put $z = \frac{x-\mu}{\sigma}$. Then z is called a **standard normal variate** and its probability density function is given by $p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \le z \le \infty$.

Mean of the standard normal distribution is 0 and variance is 1. We write $Z \sim N(0, 1)$.

3. The area under the **standard normal curve**



Normal curve

(i) between $z = -1$ and $z = 1$ is 0.6827 ($\because$ total area under the standard normal curve is 1) i.e., $p(-1 < z < 1) = 0.6827$

(ii) between $z = -2$ and $z = 2$ is 0.9545 i.e., $p(-2 < z < 2) = 0.9545$

(iii) between $z = -3$ and $z = 3$ is 0.9973 i.e., $p(-3 < z < 3) = 0.9973$

In other words, $p(\mu - \sigma < x < \mu + \sigma) = 0.6827$

$p(\mu - 2\sigma < x < \mu + 2\sigma) = 0.9545$ and

$p(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$

$p(\mu - 1.96\,\sigma < x < \mu + 1.96\,\sigma) = 0.95$

$p(\mu - 2.58\,\sigma < x < \mu + 2.58\,\sigma) = 0.99$

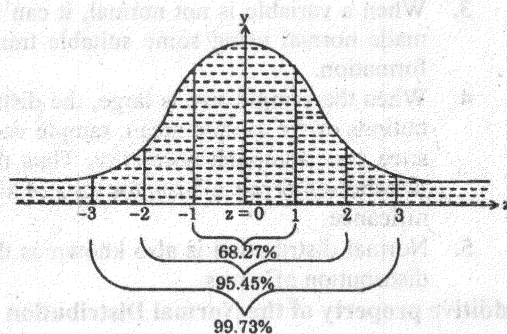
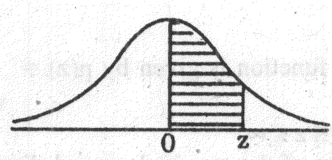


Fig. Areas under standard normal curve

4. Since the distribution is symmetric, we consider only positive values of z.



Then, $\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{z^2}{2}} dz$ gives the area under the normal curve between z = 0 and z = z.

For different values of z, $\phi(z)$ can be calculated. These values of $\phi(z)$ have been tabulated. All standard text books of statistics contain this **tables of areas**.

5. There is another table, known as the **table of ordinates**. We read $p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ corresponding to a value of z from this table. This table can also be found in any book on statistics.

Importance of the Normal Distribution

1. Many distributions tend to a normal distribution in the limit.
2. It is found that normal distribution satisfactorily describes the events arising in the fields of education, biology, sociology etc.
3. When a variable is not normal, it can be made normal using some suitable transformation.
4. When the sample size is large, the distributions of the sample mean, sample variance etc., approach normality. Thus the distribution forms a basis for tests of significance.
5. Normal distribution is also known as the distribution of errors.

Additive property of the Normal Distribution

If X and Y are two independent normal variables with means μ_1, μ_2 and variances σ_1^2, σ_2^2 respectively, then Z = X + Y is also a normal variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$

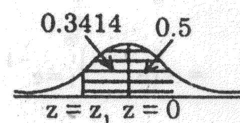
i.e., $X \sim N\left(\mu_1, \sigma_1^2\right)$,

$Y \sim N\left(\mu_2, \sigma_2^2\right)$

$\Rightarrow X + Y \sim N\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right)$

Example: If $P(Z \geq Z_1) = 0.8414$ in N (0, 1), then find Z_1.

Solution



Since $P(Z \geq Z_1) = 0.8414 \geq 0.5$, it follows that Z_1 must lie to the left of Z = 0.

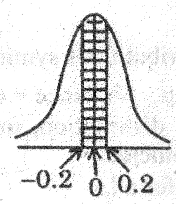
$\therefore Z_1$ is negative.

By referring to the table of areas, corresponding to Z = 1, area is 0.3413.

$\therefore Z_1 = -1$

Example: The length of 800 tubes is normally distributed with mean 66 cm and standard deviation 5 cm. Find the number of tubes with lengths lying between 65 cm and 67 cm.

Solution



Given $\mu = 66$, $\sigma = 5$

$$\therefore Z = \frac{X - \mu}{\sigma} = \frac{X - 66}{5}$$

When X = 65, $Z = \frac{65 - 66}{5} = -0.2$

When X = 67, $Z = \frac{67 - 66}{5} = 0.2$

$\therefore P(65 \leq X \leq 67) = P(-0.2 \leq Z \leq 0.2)$

= Area of the shaded part under the standard normal curve

$= 2 \times 0.0793 = 0.1586$

(By referring to the table of areas corresponding to Z = 0.2, value is 0.0793; area to the left of Z = 0 is equal to the area to the right of Z = 0 by symmetry. So we write 2×0.0793)

$\therefore$ required number of tubes = 800×0.1586 = **127**.

Example: The stores, say X, of an examination may be assumed to be a continuous random variable normally distributed with $\mu = 75$ and $\sigma^2 = 64$. What is the probability that

(a) a store chosen at random will be between 80 and 85,

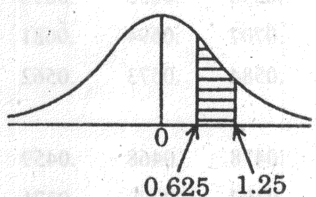
(b) a store will be greater than 85% and

(c) a store will be less than 90%?

Solution

Given $\mu = 75, \sigma = 8 \quad \therefore Z = \dfrac{X - 75}{8}$

(a) When X = 80, $Z = \dfrac{5}{8} = 0.625$



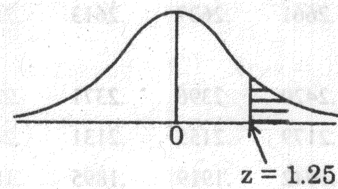
$\therefore P(80 \le X \le 85) = P(0.625 \le Z \le 1.25)$

= area of the shaded part in **Figure**

– area corresponding to Z = 12.5

= 0.3944 – 0.2324 = 0.1620

$\therefore$ there is about 16% chance of the store to lie between 80 and 85.



(b) $P(X > 85) = P(Z > 1.25)$

= area of the shaded part

= 0.5 – 0.3944

= 0.1056

(c) $P(X < 90) = P(Z < 1.875)$

($\because$ When X = 90, Z = 1.875)

= area of the shaded part

= 0.5 + 0.4693 (nearly)

= 0.9693 nearly

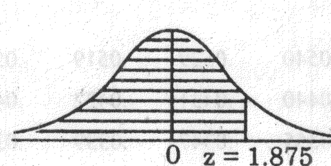
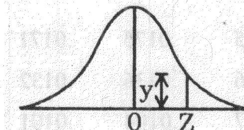


Table 1:

Ordinates (y) of the Standard Normal Curve at z



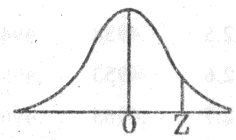
Z	0	1	2	3	4	5	6	7	8	9
0.0	.3989	.3989	.3989	.3988	.3986	.3984	.3982	.3980	.3977	.3973
0.1	.3970	.3965	.3961	.3956	.3951	.3945	.3939	.3932	.3925	.3918
0.2	.3910	.3902	.3894	.3885	.3876	.3867	.3857	.3847	.3836	.3825
0.3	.3814	.3802	.3790	.3778	.3765	.3752	.3739	.3725	.3712	.3697
0.4	.3683	.3668	.3653	.3637	.3621	.3605	.3589	.3572	.3555	.3538

Z	0	1	2	3	4	5	6	7	8	9
0.5	.3521	.3603	.3485	.3467	.3448	.3429	.3410	.3391	.3372	.3352
0.6	.3332	.3312	.3292	.3271	.3251	.3230	.3209	.3187	.3166	.3144
0.7	.3123	.3101	.3079	.3056	.3034	.3011	.2989	.2966	.2943	.2920
0.8	.2897	.2874	.2850	.2827	.2803	.2780	.2756	.2732	.2709	.2685
0.9	.2661	.2637	.2613	.2589	.2565	.2541	.2516	.2492	.2468	.2444
1.0	.2420	.2396	.2371	.2347	.2323	.2299	.2275	.2251	.2227	.2203
1.1	.2179	.2155	.2131	.2107	.2083	.2059	.2036	.2012	.1989	.1966
1.2	.1942	.1919	.1895	.1872	.1849	.1826	.1804	.1781	.1756	.1736
1.3	.1714	.1691	.1669	.1647	.1626	.1604	.1582	.1561	.1539	.1518
1.4	.1497	.1476	.1456	.1435	.1415	.1394	.1374	.1354	.1334	.1315
1.5	.1295	.1276	.1257	.1238	.1219	.1200	.1182	.1163	.1145	.1127
1.6	.1109	.1092	.1074	.1057	.1040	.1023	.1006	.0989	.0973	.0957
1.7	.0940	.0925	.0909	.0893	.0878	.0863	.0848	.0833	.0818	.0804
1.8	.0790	.0775	.0761	.0748	.0734	.0721	.0707	.0694	.0681	.0605
1.9	.0656	.0644	.0632	.0620	.0608	.0596	.0584	.0573	.0562	.0551
2.0	.0540	.0529	.0519	.0508	.0498	.0488	.0478	.0468	.0459	.0449
2.1	.0440	.0431	.0422	.0413	.0404	.0396	.0387	.0379	.0371	.0363
2.2	.0355	.0347	.0339	.0332	.0325	.0317	.0310	.0303	.0297	.0290
2.3	.0283	.0277	.0270	.0264	.0258	.0252	.0246	.0241	.0235	.0229
2.4	.0224	.0219	.0213	.0208	.0203	.0198	.0194	.0189	.0184	.0180
2.5	.0175	.0171	.0167	.0163	.0158	.0154	.0151	.0147	.0143	.0139
2.6	.0136	.0132	.0129	.0126	.0122	.0119	.0116	.0113	.0110	.0107
2.7	.0104	.0101	.0099	.0096	.0093	.0091	.0088	.0086	.0084	.0081
2.8	.0079	.0077	.0075	.0073	.0071	.0069	.0067	.0065	.0063	.0061
2.9	.0060	.0058	.0056	.0055	.0053	.0051	.0050	.0048	.0047	.0046
3.0	.0044	.0043	.0042	.0040	.0039	.0038	.0037	.0036	.0035	.0034
3.1	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026	.0025	.0025
3.2	.0024	.0023	.0022	.0022	.0021	.0020	.0020	.0019	.0018	.0018

Z	0	1	2	3	4	5	6	7	8	9
3.3	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014	.0013	.0013
3.4	.0012	.0012	.0012	.0011	.0011	.0010	.0010	.0010	.0009	.0009
3.5	.0009	.0008	.0008	.0008	.0008	.0007	.0007	.0007	.0007	.0006
3.6	.0006	.0006	.0006	.0005	.0005	.0005	.0005	.0005	.0005	.0004
3.7	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003	.0003
3.8	.0003	.0003	.0003	.0003	.0003	.0002	.0002	.0002	.0002	.0002
3.9	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001

Table 2:

Areas under the Standard Normal Curve from 0 to Z



Z	0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0567	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3503	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4177
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545

1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4918	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4994	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

OBJECTIVE QUESTIONS

1. An experiment succeeds twice as often as it fails. Then the probability that there will be at least 4 successes in 6 trials is

(a) $\frac{496}{729}$ (b) $\frac{233}{729}$

(c) $\frac{496}{233}$ (d) none of these

2. If the probability of a male birth is $\frac{1}{2}$, the probability that out of 3 children born to a woman, two are males and one is a female is

(a) $\frac{1}{8}$ (b) $\frac{1}{2}$

(c) $\frac{3}{8}$ (d) none of these

3. If X is a random variable with a Binomial distribution with n = 5, p = 0.2 then the most probable value (value taken with the highest probability) of X is

(a) 1 (b) 2

(c) 3 (d) 4

4. Which of the following statements are not correct?

The limiting form of the binomial distribution is

(i) the poisson distribution

(ii) the normal distribution

(iii) the uniform distribution

(iv) the exponential distribution

(a) (i), (ii) only (b) (ii), (iii) only

(c) (iii), (iv) only (d) all the four

5. Match List I (Distribution) with List II (Situation) and select the correct answer using the codes given below the Lists.

List I	List II
A. Exponential	1. Rounding off error
B. Poisson	2. Life-length of electronic component
C. Uniform	3. Printing errors of spelling per page in a book
D. Normal	4. Events in Biology

Codes

	A	B	C	D
(a)	3	2	4	1
(b)	2	3	1	4
(c)	1	3	2	4
(d)	1	2	3	4

6. If four coins are tossed together the chance that there are 2 heads and 2 tails is

(a) $\frac{3}{8}$ (b) $\frac{5}{8}$

(c) $\frac{1}{8}$ (d) $\frac{3}{4}$

7. In a Binomial distribution

(a) mean = variance

(b) mean < variance

(c) mean > variance

(d) no relationship exists between mean and variance.

8. 8 coins are thrown simultaneously 512 times. Then the number of times at least 6 heads will show up is

(a) 74 (b) 37

(c) 56 (d) 18

9. A student, unprepared for a quiz, takes a six-question true - false quiz and guesses each answer. The probability that the student will come out successful in the quiz test if the passing grade is at least four correct answers.

(a) $\frac{1}{4}$ (b) $\frac{5}{32}$

(c) $\frac{3}{4}$ (d) none of these

10. The mean and variance of a Binomial distribution are 16 and 12 respectively. Then $P(X \geq 2)$ is

(a) $\left(\frac{3}{4}\right)^{64}$ (b) $\frac{3^{63}\cdot 67}{4^{64}}$

(c) $1-\frac{3^{63}\cdot 67}{4^{64}}$ (d) none of these

11. In a town A, 20% of the population is literate. 200 investigators take a sample of 10 each to find out whether they are literate or not. Then the number of investigators who are expected to report that three people or less are literate in the samples is

(a) $200 \sum_{x=0}^{3} {}^{10}C_x \left(\frac{4}{5}\right)^{10-x}$

(b) $200 \sum_{x=0}^{3} {}^{10}C_x \frac{4^{10-x}}{5^{10}}$

(c) $200 \sum_{x=0}^{3} {}^{10}C_x \left(\frac{4}{5}\right)$

(d) none of these

12. For a B (n, p) mean is 16 and variance is 12. Then the third central moment is
(a) 4 (b) 6
(c) 8 (d) none of these

13. X and Y are independent Binomial variables following B $\left(5, \frac{1}{2}\right)$ and B $\left(7, \frac{1}{2}\right)$. Then P (x + y = 3) is
(a) $\frac{55}{1024}$ (b) $\frac{55}{512}$
(c) ${}^{12}C_3 \times \frac{1}{2^{11}}$ (d) none of these

14. A radar unit tracking a target detects it with probability p during a surveillance cycle. The detection of the target in each cycle is independent of other cycles. The probability that the target will be detected within the first n cycles is
(a) p^n (b) $1-(1-p)^n$
(c) $\sum_{k=1}^{n} p^k$ (d) $(1-p)^n$

15. If X is a Poisson variate such that P (x = 2) = 9 P (x = 4) + 90 P (x = 6) then the mean of the distribution is
(a) 2 (b) 1
(c) 3 (d) 4

16. In a Poisson distribution concerning accident statistics, the average number of accidents is two. Then the probability that there will be four accidents in a week is
(a) $1-\frac{4}{15e^2}$ (b) $\frac{1}{4-e^2}$
(c) $\frac{2}{3e^2}$ (d) $\frac{1}{e^2}$

17. A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Then the number of days in April, on which only one car is used is (given $e^{-1.5} = 0.22$)
(a) 8 (b) 10
(c) 13 (d) 15

18. A manufacturer of medicine bottles finds that 0.1 % of the bottles are defective. The bottles are packed in boxes of 500 bottles. If 100 boxes are brought then the number of boxes with two or more defectives is (given $e^{-0.5} = 0.6065$)
(a) 9 (b) 61
(c) 19 (d) 91

19. The probability that a Poisson variate X takes a positive value is $(1 - e^{-1.5})$. Then the probability that X lies between − 1.5 and 1.5 is
(a) $1.5\ e^{-1.5}$ (b) $e^{-1.5}$
(c) $2.5\ e^{-1.5}$ (d) none of these

20. If X and Y are independent Poisson variates such that P (X = 1) = P (X = 2) and P (Y = 2) = P (Y = 3) then the variance of x − 2y is
(a) 12 (b) 8
(c) 10 (d) 14

21. If X is a Poisson variable with mean m then the expectation of e^{-kx} where k is a constant is
(a) $e^{m(1-e^{-k})}$ (b) $e^{-m(e^{-k}+1)}$
(c) $e^{m(1-e^{k})}$ (d) $e^{-m(1-e^{-k})}$

22. A random variable X has the density function

$$f(x)=\begin{cases} c\,e^{-3x}, & x>0 \\ 0, & x \le e\,0 \end{cases}$$

Then the value of the constant c is
(a) 3 (b) 1
(c) $\frac{1}{3}$ (d) 9

23. A random variable X has the density function e^{-x} for $x \geq 0$. Then $P(|X-1| \geq 2)$ equals

(a) e^{-2} (b) e^{-1}
(c) e^{-3} (d) e^{-4}

24. If X is uniformly distributed in $-2 \leq x \leq 2$ then $P(X < 1)$ is

(a) $\frac{3}{4}$ (b) $\frac{1}{4}$
(c) $\frac{1}{2}$ (d) 0

25. If X is a continuous random variable whose distribution is constant in an interval $a \leq x \leq b$ and 0 elsewhere then the mean value of X is

(a) $\frac{a+b}{2}$ (b) $\frac{b-a}{2}$
(c) $\frac{a+2b}{2}$ (d) $\frac{2a+b}{2}$

26. If X is uniformly distributed over (0, 10) then $P(1 < x < 6)$ equals

(a) $\frac{1}{2}$ (b) $\frac{3}{10}$
(c) $\frac{4}{7}$ (d) $\frac{5}{7}$

27. If $X \sim \cup(-a, a)$ (that is, X follows a uniform distribution over $(-a, a)$) then the value of a such that $P(|X|>2)=\frac{3}{4}$ is

(a) 6 (b) 8
(c) 5 (d) 2

28. Life of electric bulbs follows an exponential distribution with a mean of 1500 hours. The probability that the bulbs will have less than 1000 hours is

(a) $\frac{1}{e^{2/3}-1}$ (b) $\frac{1}{1+e^{2/3}}$
(c) $e^{-\frac{2}{3}}$ (d) $1-e^{-\frac{2}{3}}$

29. Match List I with List II and choose the correct answer using the codes given below the Lists.

List I (Distribution)	List II (Property)
A. Binomial	1. Mean $=\frac{1}{2}$, Variance $=\frac{1}{4}$
B. Poisson	2. Mean $=\frac{7}{2}$, Variance $=\frac{3}{4}$
C. Exponential	3. Mean = 12, Variance =3
D. Uniform	4. Mean = 1, Variance = 1

Codes

	A	B	C	D
(a)	3	4	2	1
(b)	3	4	1	2
(c)	4	3	2	1
(d)	4	3	1	2

30. Which one of the following statements is true?

1. If X and Y have standard normal distributions then their sum X + Y also has a standard normal distribution.
2. If X and Y have Poisson distribution with not necessarily the same parameter then their sum X + Y also has a Poisson distribution.

(a) 1 only (b) 2 only
(c) both 1 and 2 (d) neither 1 nor 2

31. If X is normally distributed with mean 5 and standard deviation 2, then $P(X > 7)$ equals

(a) 0.4772 (b) 0.0228
(c) 0.3413 (d) 0.1587

32. The mean weight of 100 students at a certain college is 75 kg and the standard deviation is 7 kg. Assuming that the weights are normally distributed the number of students weighing between 61 kg and 89 kg is approximately

(a) 90 (b) 95
(c) 99 (d) 68

33. The income distribution of workers in a factory was found to be normal with mean of Rs. 500 and standard deviation equal to Rs. 50. There are 228 persons getting above

Rs. 600. Then the total strength of the factory is (area under the normal curve between 0 and 2 is 0.4772)

(a) 20000 (b) 1000
(c) 5000 (d) 10000

34. The distribution of monthly salaries of 3000 teachers conforms to a normal curve with an average monthly salary of Rs. 600 and a standard deviation of Rs. 100. Then the number of teachers drawing over Rs. 800 is

(a) 68 (b) 90
(c) 160 (d) 78

35. The scores made by 500 candidates in a certain test are normally distributed with mean 500 and Standard Deviation 100. Then the number of candidates scoring less than 400 is

(a) 110 (b) 95
(c) 60 (d) none of these

36. Among 600 candidates of GATE Computer Science and Engineering in a city A, the mean mark is 80.75 with standard deviation 5. The number of candidates scoring over 89 is approximately

(a) 30 (b) 15
(c) 60 (d) 10

37. A normal distribution has mean 6 and standard deviation 3. The probability that an observation taken randomly will have a negative value is

(a) 0.05 (b) 0.1
(c) 0.3 (d) 0.0228

38. 15000 students appeared for an examination. The mean marks were 9 and the Standard deviation is 6. Assume the marks to be normally distributed. If the top 2.5% got grade A then the minimum marks for grade A is

(a) 67 (b) 65
(c) 61 (d) 59

39. Two independent random variables X and Y are both normally distributed with means 1 and 2 and standard deviations 3 and 4 respectively. If $V = X - Y$ then $P(V + 1 \le 0)$ is

(a) 0.50 (b) 0.60
(c) 0.25 (d) 0.00

40. Which of the following statements is/are false?

1. Binomial distribution is symmetrical if $p = q = \frac{1}{2}$
2. For a normal distribution coefficient of skewness is zero.

(a) 1 only (b) 2 only
(c) both 1 and 2 (d) neither 1 nor 2

KEY

1. (a)	2. (c)	3. (a)	4. (c)
5. (b)	6. (a)	7. (c)	8. (a)
9. (d)	10. (c)	11. (b)	12. (b)
13. (a)	14. (b)	15. (b)	16. (c)
17. (b)	18. (a)	19. (c)	20. (d)
21. (d)	22. (a)	23. (c)	24. (a)
25. (a)	26. (a)	27. (b)	28. (d)
29. (b)	30. (b)	31. (d)	32. (b)
33. (d)	34. (a)	35. (d)	36. (a)
37. (d)	38. (c)	39. (a)	40. (d)

EXPLANATORY NOTES

1. Probability of success $= p = \frac{2}{3}$ ($\because$ when there are 3 trials of the experiment, success is in 2 trials and failure in 1 trial)

$$\therefore\ q = 1 - p = \frac{1}{3} \text{ and } n = 6$$

$$\text{So } P(X = x) = {}^nC_x\, p^x\, q^{n-x}$$

$$= {}^6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}$$

$\therefore$ required probability $= P(X \ge 4)$

$= P(X = 4) + P(X = 5) + P(X = 6)$

$$= {}^6C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 + {}^6C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right) + {}^6C_6\left(\frac{2}{3}\right)^6$$

$$= \frac{1}{3^6}(15\times2^4 + 6\times2^5 + 2^6)$$

$$= \frac{496}{729}$$

2. p = probability of a male birth $=\frac{1}{2}\therefore q=\frac{1}{2}$
n = 3.

$$\therefore P(X=x)={}^{n}C_x p^x q^{n-x}={}^{3}C_x\left(\frac{1}{2}\right)^3$$

So required probability

$$=P(X=2)={}^{3}C_2\left(\frac{1}{2}\right)^3=\frac{3}{8}$$

3. $P(X=x)={}^5C_x(0.2)^x(0.8)^{5-x}$

$$\therefore P(X=0)={}^5C_0(0.2)^0(0.8)^5=(0.8)^5=0.32768$$

$P(X=1)={}^5C_1(0.2)(0.8)^4=0.4096$

$P(X=2)={}^5C_2(0.2)^2(0.8)^3=0.2048$

$P(X=3)={}^5C_3(0.2)^3(0.8)^2=0.0512$

$P(X=4)={}^5C_4(0.2)^4(0.8)=0.0064$

$P(X=5)={}^5C_5(0.2)^5=0.00032$

6. Required probability $={}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=\frac{6}{16}=\frac{3}{8}$

7. In a Binomial distribution, mean = np, variance = npq. Since q < 1, npq < np. Or mean > variance.

8. $P(X\geq 6)=P(X=6)+P(x=7)+P(X=8)$

$$={}^8C_6\left(\frac{1}{2}\right)^8+{}^8C_7\left(\frac{1}{2}\right)^8+{}^8C_8\left(\frac{1}{2}\right)^8$$

$$=\frac{1}{256}({}^8C_6+{}^8C_7+1)$$

$$=\frac{1}{256}(28+8+1)=\frac{37}{256}$$

Required number of times

$$=512\times\frac{37}{256}=74$$

9. Here $n=6,\ p=\frac{1}{2}=q$

$P(X\geq 4)=P(X=4)+P(X=5)+P(X=6)$

$$={}^6C_4\left(\frac{1}{2}\right)^6+{}^6C_5\left(\frac{1}{2}\right)^6+{}^6C_6\left(\frac{1}{2}\right)^6$$

$$=\frac{1}{64}(15+6+1)=\frac{11}{32}$$

10. Given npq = 12, np = 16

$$\therefore q=\frac{12}{16}=\frac{3}{4}$$

$p=1-q=\frac{1}{4}$. So n = 64

Now $P[X\geq 2]=1-P(X<2)$

$$=1-[P(X=0)+P(X=1)]$$

$$=1-[{}^{64}C_0p^0q^{64}+{}^{64}C_1pq^{63}]$$

$$=1-\left(\frac{3}{4}\right)^{63}\left(\frac{3}{4}+16\right)$$

$$=1-\frac{3^{63}\cdot 67}{4^{64}}$$

11. Given N = 200, n = 10, $p=\frac{20}{100}=\frac{1}{5},q=\frac{4}{5}$

$P(X\leq 3)=P(X=0)+P(X=1)+P(X=2)+P(X=3)$

$$=\sum_{x=0}^{3}{}^{10}C_x p^x q^{10-x}$$

$\therefore$ number of investigators

$$=N\sum_{x=0}^{3}{}^{10}C_x\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{10-x}$$

$$=200\sum_{x=0}^{3}{}^{10}C_x\frac{4^{10-x}}{5^{10}}$$

12. Third central moment $=npq(q-p)$

$$=16\times\frac{3}{4}\left(\frac{3}{4}-\frac{1}{4}\right)$$

$$\left(\because \frac{npq}{np}=\frac{12}{16}\text{ gives } q=\frac{3}{4},p=\frac{1}{4}\right)$$

$$=6$$

13. $P(X=x)={}^5C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{5-x}={}^5C_x\left(\frac{1}{2}\right)^5$

$P(Y=y)={}^5C_y\left(\frac{1}{2}\right)^y\left(\frac{1}{2}\right)^{7-y}={}^5C_y\left(\frac{1}{2}\right)^7$

$\therefore$ required probability = P (X = 0, Y = 3) + P (X = 1, Y = 2) + P (X = 2, Y = 1) + P (X = 3, Y = 0)

$$= {}^5C_0\left(\frac{1}{2}\right)^5 \times {}^7C_3\left(\frac{1}{2}\right)^7 + {}^5C_1\left(\frac{1}{2}\right)^5$$

$$\times {}^7C_2\left(\frac{1}{2}\right)^7 + {}^5C_2\left(\frac{1}{2}\right)^5 \times {}^7C_1\left(\frac{1}{2}\right)^7$$

$$+ {}^5C_3\left(\frac{1}{2}\right)^5 \times {}^7C_0\left(\frac{1}{2}\right)^7$$

$$= \left(\frac{1}{2}\right)^{12} [{}^5C_0 \times {}^7C_3 + {}^5C_1 \times {}^7C_2$$

$$+ {}^5C_2 \times {}^7C_1 + {}^5C_3 \times {}^7C_0]$$

$$= \frac{1}{2^{12}}(35+105+70+10) = \frac{220}{2^{12}} = \frac{55}{1024}$$

14. Probability that the target will not be detected within the first n cycles $= (1-p)^n$

$\therefore$ required probability $= 1-(1-p)^n$

15. Let λ be the mean.

Given $e^{-\lambda}\frac{\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$

i.e., $\frac{1}{2} = \frac{9\lambda^2}{24} + \frac{90\lambda^4}{720}$

or $\lambda^4 + 3\lambda^2 - 4 = 0$

i.e., $(\lambda^2+4)(\lambda^2-1) = 0$

$\therefore \lambda^2 = -4$ which is impossible

and $\lambda = \pm 1$; $\lambda = -1$ is impossible.

16. Given $\lambda = 2$

$$\therefore P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2}2^x}{x!}, x = 0,1,2...$$

So $P(X=4) = \frac{e^{-2}2^4}{4!} = \frac{2}{3e^2}$

17. Let X = number of demands each day. Then X is a random variable taking only three values 0, 1, 2. If there is demand for more than two cars, then the firm has to refuse.

$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$ where $\lambda = 1.5$

$$= \frac{e^{-1.5}(1.5)^x}{x!}$$

$P(X=1) = e^{-1.5}(1.5) = 0.22 \times 1.5$

So the number of days in April on which only one car is used $= 30 \times P(X=1)$

$= 30 \times 0.22 \times 1.5 \simeq 10$

18. $\lambda = np = \frac{500 \times 0.1}{100} = 0.5$

$$\therefore P(X=x) = \frac{e^{-0.5}(0.5)^x}{x!}$$

$P(X \geq 2) = 1 - P(X<2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - e^{-0.5}(1+0.5)$

$= 1 - 0.6065 \times 1.5$

$= 0.09025$

Then number of boxes with $P(X \geq 2)$

$= 100 \times 0.09025 = 9$ (nearly)

19. $P(X>0) = 1 - P(X \leq 0)$

$= 1 - P(X=0)$

$= 1 - e^{-\lambda} = 1 - e^{-1.5}$ given

$\therefore \lambda = 1.5$

So $P(-1.5 \leq x \leq 1.5) = P(X=0) + P(X=1)$

$= e^{-\lambda} + e^{-\lambda}\lambda = (1+\lambda)e^{-\lambda}$

$= 2.5\,e^{-1.5}$

20. $P(X=1) = P(X=2)$

$$\Rightarrow e^{-\lambda}\lambda = \frac{e^{-\lambda}\lambda^2}{2!}$$

$\lambda = 2$

$\therefore$ var $X = \lambda = 2$

$P(Y=2) = P(Y=3) \Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-\lambda}\lambda^3}{3!}$

$\Rightarrow \lambda = 3$

$\therefore$ var $Y = \lambda = 3$

Now var $(X-2Y)$ = var X + 4 var Y

$= 2 + 12 = 14$

21. $E(e^{-kx}) = \sum_{x=0}^{\infty} e^{-kx} P(X=x)$

$$= \sum_{x=0}^{\infty} e^{-kx}\frac{e^{-m}m^x}{x!}$$

$$= e^{-m}\sum_{x=0}^{\infty}\frac{(me^{-k})^x}{x!}$$

$$= e^{-m} \cdot e^{m e^{-k}}$$

$$= e^{-m(1-e^{-k})}$$

22. Since f(x) is a density function,

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} C\,e^{-3x}\,dx$$

$$= \left(\frac{C\,e^{-3x}}{-3}\right)_0^{\infty} = \frac{C}{3}$$

$\therefore$ C = 3

23. $f(x)=\begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

$|x-1| \geq 2 \Rightarrow x \geq 3$ or $x \leq -1$

$\therefore P(|X-1| \geq 2) = P(X \geq 3) + P(X \leq -1)$

$$= \int_{3}^{\infty} e^{-x}\,dx + \int_{-\infty}^{-1} 0\,dx$$

$$= e^{-3}$$

24. $f(x)=\begin{cases} \frac{1}{4}, & -2 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

$$\therefore P(x<1) = \int_{-2}^{1} f(x)\,dx = \int_{-2}^{1} \frac{1}{4}\,dx = \frac{3}{4}$$

25. $f(x)=\begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{elsewhere} \end{cases}$

Mean = E (X)

$$= \int_{a}^{b} x\,f(x)\,dx$$

$$= \int_{a}^{b} \frac{x}{b-a}\,dx = \frac{1}{2(b-a)}(x^2)_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

26. Here $f(x) = \begin{cases} \frac{1}{10}, & 0 < x < 10 \\ 0, & \text{otherwise} \end{cases}$

$\therefore P(1 < X < 6)$

$$= \int_{1}^{6} f(x)\,dx = \int_{1}^{6} \frac{1}{10}\,dx = \frac{1}{2}$$

27. $f(x)=\begin{cases} \frac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

$|x| > 2 \Rightarrow x > 2$ or $x < -2$

$$\therefore P(|x|>2) = \frac{3}{4} \Rightarrow P(x>2)+P(x<-2) = \frac{3}{4}$$

$$\Rightarrow \int_{2}^{a} \frac{1}{2a}\,dx + \int_{-a}^{-2} \frac{1}{2a}\,dx = \frac{3}{4}$$

$$\Rightarrow \frac{1}{2a}(a-2) + \frac{1}{2a}(-2+a) = \frac{3}{4}$$

$$\Rightarrow \frac{a-2}{a} = \frac{3}{4}$$

$$\Rightarrow a = 8$$

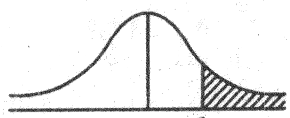
28. Mean $= \frac{1}{a} = 1500 \quad \therefore a = \frac{1}{1500}$

$$f(x)=\begin{cases} \frac{1}{1500} e^{-\frac{x}{1500}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X<1000) = \int_{0}^{1000} \frac{1}{1500} e^{-\frac{x}{1500}}\,dx = 1 - e^{-\frac{2}{3}}$$

31. $Z = \frac{7-5}{2} = 1$

From table of areas under normal curve, required probability = 0.5 – 0.3413 = 0.1587



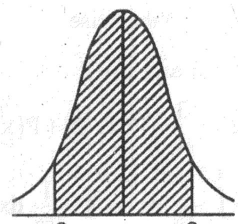
32. $Z = \dfrac{X - 75}{7}$

When $X = 61, Z = \dfrac{61 - 75}{7} = -2$

When $X = 89, Z = \dfrac{89 - 75}{7} = 2$

Area under the normal curve between Z = – 2 and Z = 2 is $2 \times 0.4772 = 0.9544$.

$\therefore$ number of students $= 100 \times 0.9544$
$= 95$ (nearly)

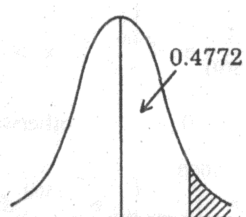


33. $Z = \dfrac{X - 500}{50}$

When $X = 600, Z = \dfrac{600 - 500}{50} = 2$

Area of the shaded part $= 0.5 - 0.4772$
$= 0.0228$

Given $N \times 0.0228 = 228 \therefore N = 10000$

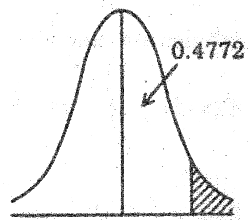


34. $Z = \dfrac{X - 600}{100}$

When X = 800, Z = 2

Area of the shaded part = 0.5 – 0.4772
= 0.0228

$\therefore$ number of teachers satisfying the condition
$= 3000 \times 0.0228 = 68$

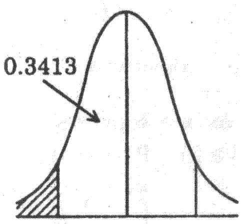


35. $Z = \dfrac{X - 500}{100}$

When X = 400, Z = – 1

Area of the shaded part = 0.5 – 0.3413
= 0.1587

$\therefore$ number of candidates $= 500 \times 0.1587 \simeq 79$

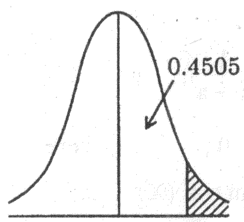


36. $Z = \dfrac{X - 80.75}{5}$

When $X = 89, Z = \dfrac{8.25}{5} = 1.65$

Area of the shaded part
= 0.5 – 0.4505 (from tables)
= 0.0495

Number of candidates satisfying the condition

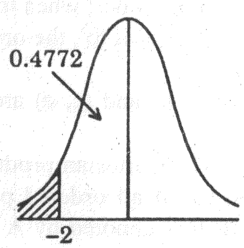


$= 600 \times 0.0495 \simeq 30$

37. $Z = \frac{X - 6}{3}$

$X < 0$ means $Z < -2$

Area of the shaded part $= 0.5 - 0.4772$
$= 0.0228$

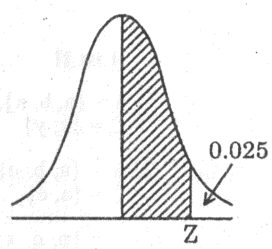


38. $Z = \frac{X - 49}{6}$

Area of the shaded part $= 0.5 - 0.025$
(from data given)
$= 0.475$

Corresponding value for Z from table of areas = 1.96

$\therefore 1.96 = \frac{X - 49}{6}$ or $X = 49 + 6 \times 1.96 \simeq 61$

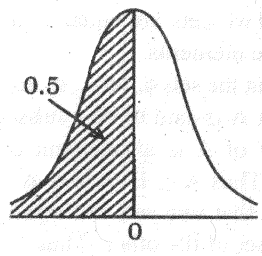


39. Since X, Y are independent and normal,
$V = X - Y$ is normal with mean $1 - 2 = -1$ and variance $9 + 16 = 25$.

$\therefore$ Mean of $V = -1$ and Standard deviation of $V = 5$.

Now $Z = \frac{V + 1}{5}$ $\therefore P(V + 1 \le 0) = P(Z \le 0)$

= area of the shaded part
= 0.5



3. DISCRETE MATHEMATICS

3.1 SETS

Introduction

Set: A **set** is intuitively understood as a well-defined collection of **objects.** The objects are called the **elements** or members of the set; we also say that the objects belong to the set. A set has to be conceived as a whole.

The objects of a set can be anything (numbers, rivers etc.). They have to be **definite** and **distinguishable**. That is **(i)** given a set and an object we must be able to say definitely whether the object belongs to the set or not and **(ii)** also we must be able to distinguish between one object and another.

An element x belonging to a set A will be denoted by $x \in A$. So $x \notin A$ means x is not a member of A i.e., $x \notin A \equiv \rceil(x \in A)$.

If $A = \{0, 2, 4, 6, 8 \ldots\}$ then we can write $A = \{x | x \text{ is an even integer} \ge 0\}$.

Empty set: A set containing no elements is called an **empty set** or **null set** or **void set**. It is denoted by ϕ.

Singleton set: A set consisting of only one element is called a **singleton set.**

Thus if $A = \{x \mid 5x - 3 = 0, x \text{ is an integer}\}$ then

$A = \phi$ ($\because x = \frac{3}{5}$ which is not an integer).

If $B = \{x \mid x + 8 = 9\}$ then $B = \{1\}$.

Finite and Infinite sets: A set A is said to be **finite** if either $A = \phi$ or there exists a natural number n such that A has n elements. A set which is not finite is said to be **infinite.**

$C = \{1, 3, 5, 10\}$ is a finite set whereas

$I = \{1, 2, 3, 4, ...\}$ is an infinite set.

Equal sets: Two sets are equal if and only if they have the same elements.

Observe that the sets ϕ, $\{\phi\}$, $\{0\}$ are unequal.

Subset: A set A is said to be a **subset** of a set B, if each element of A is an element of B. We write then $A \subseteq B$. Thus $A \subseteq B \Leftrightarrow x \in A \longrightarrow x \in B$. It is immediate that two sets are equal if and only if each is a subset of the other. Thus,

$A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Power set: For any set A, the family of all subsets of A is called the power set P(A) is of A. If A has n elements then P(A) has 2^n elements.

Universal set: In any mathematical discussion all the sets under consideration are likely to be subsets of a non-empty fixed set known as the **universal set E**.

Proper subset: A set A is a **proper subset** of a set B, if $A \subseteq B$ and $A \neq B$. We write $A \subset B$.

Indexed set: Let I be a given set. For each $i \in I$, let there be associated a set S_i. Then the set I is called the **index set** and the sets S_i, S_j, ... are called the **indexed sets.**

Union of sets A, B: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
$= \{x \mid x \in A \vee x \in B\}$

Intersection of sets A, B:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} = \{x \mid x \in A \wedge x \in B\}$$

Difference or Relative complement:

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Complement of a set: Let E be the universal set. Then $E - A$ denoted by $\bar{A}$ or A' is called the **complement** of A.

Thus $\bar{A} = \{x \mid x \in E \wedge x \notin A\} = E - A$.

Observe that $A - B = A \cap \bar{B} = A - (A \cap B)$

Symmetric difference or Boolean sum

$A \Delta B = (A - B) \cup (B - A)$

$= \{x \mid x \in A \ \bar{\vee}\ x \in B\} = A \oplus B$

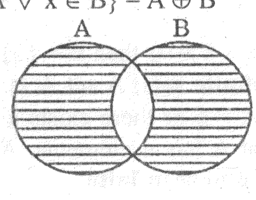


Fig. Venn diagram for $A \oplus B$

It can be easily shown that,

$$A \Delta B = B \Delta A$$
$$A \Delta \phi = \phi \Delta A = A$$
$$A \Delta A = \phi \text{ and } A \Delta B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$$

Ordered pair: An ordered pair consists of two objects a and b in a given order; when the first co-ordinate is 'a' and the second is 'b', the ordered pair is denoted by (a, b).

Two ordered pairs (a, b) and (c, d) are equal iff $a = c$ and $b = d$.

Cartesian product: The Cartesian product of two sets A and B is the set of all ordered pairs (a, b) where $a \in A$, $b \in B$. It is denoted by $A \times B$ (read as A cross B). Thus

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

The Cartesian product of n sets $A_1, A_2, ... A_n$ is $A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) | a_i \in A_i, i = 1, 2, ..., n\}$.

$(a_1, a_2, ... a_n)$ is called an **n-tuple.**

Remark (i): If A contains m elements and B contains n elements then $A \times B$ has mn elements.

(ii): $A \times B = B \times A \Leftrightarrow A = \phi$ or $B = \phi$ or $A = B$

Problem: Match List I (sets) with List II (Examples)

List I	List II
A. $A \cup B \subset A \cup C$ but $B \not\subseteq C$	1. $A = \{a, b, x\}$, $B = \{y\}$, $C = \{x, y\}$
B. $A \cap B \subset A \cap C$ but $B \not\subseteq C$	2. $A = \{a, b, c\}$, $B = \{a, d\}$, $C = \{a, e\}$
C. $A \cup B = A \cup C$ but $B \neq C$	3. $A = \{p, q, x\}$, $B = \{x, r\}$, $C = \{y, r\}$
D. $A \cap B = A \cap C$ but $B \neq C$	4. $A = \{a, y, z\}$, $B = \{x, z\}$, $C = \{q, y, z\}$

Codes

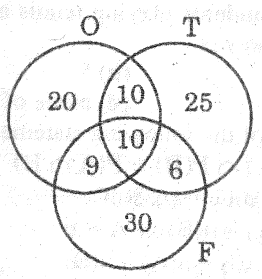
	A	B	C	D
(a)	3	4	2	1
(b)	3	4	1	2
(c)	4	3	2	1
(d)	4	3	1	2

Problem: A person was asked to conduct an enquiry into the reading habit of the magazines Outlook, Time and Femina by 100 people. He presented the following report about the number of people who read the various magazines:

All the three magazines 10, Femina but not Time 39, Outlook only 20, Femina and Outlook 19, Time 51, Femina and Time but not Outlook 6, Time and Outlook 20.

The report was dismissed as baseless. Why?

Solution: Let O, T and F denote the sets of people reading Outlook, Time and Femina respectively. In the Venn diagram, marking the number of people in the respective region we have **Figure** for



the report. Counting the number of people in the various regions, we get

$20 + 10 + 25 + 6 + 30 + 9 + 10 = 110$

But the total number interviewed is only 100.

Hence the report is a fabricated one.

Problem: Decide if the following are true or false. If true mark T and if false, F.

(a) $a \subseteq \{a, b\}$
(b) $\{a\} \subseteq \{a, b\}$
(c) $a = \{a\}$
(d) $\{3\} \subseteq \{\{1, 2\}, 4, \{2, 3\}\}$
(e) $\phi \subseteq \{\phi\}$
(f) $\phi \in \{\phi\}$
(g) $\{\phi\} \subseteq \phi$
(h) Every set has a proper subset
(i) $A \in P(A)$
(j) $A \subseteq P(A)$
(k) The set of all rationals between 0 and 1 is finite
(l) Every subset of the infinite set is infinite

Solution: **(a)** F **(b)** T **(c)** F **(d)** F **(e)** T, since empty set is a subset of each set. **(f)** T **(g)** F, since $\{\phi\}$ has one element ϕ and ϕ has no element. **(h)** F, since null set does not have any proper subset **(i)** T **(j)** F **(k)** F **(l)** F, since $\{1, 2\}$ is a finite subset of the infinite set $\{1, 2, 3, ...\}$.

Problem: What can you say about P and Q if

(a) $P \cap Q = P$ **(b)** $P \cup Q = P$
(c) $P \oplus Q = P$ **(d)** $P \cap Q = P \cup Q$?

Solution

(a) $P \subseteq Q$ **(b)** $Q \subseteq P$
(c) $Q = \phi$ **(d)** $P = Q$.

Problem: If $A = \{a, b, \{a, b\}, \phi\}$ find

(a) $A - \{a\}$ **(b)** $A - \phi$
(c) $A - \{a, b\}$ **(d)** $A - \{a, c\}$

For example, $A - \{a\} = \{a, b, \{a, b\}, \phi\} - \{a\}$
$= \{b, \{a, b\}, \phi\}$

Solution

(a) $\{b, \{a, b\}, \phi\}$ **(b)** A
(c) $\{\{a, b\}, \phi\}$ **(d)** $\{b, \{a, b\}, \phi\}$

Problem: If $A = \{a, b, c\}$ and $B = \{1, 2\}$ compute $A \times B$ and $B \times A$. Are they equal?

Solution

$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$
$B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$
$\therefore A \times B \neq B \times A$

OBJECTIVE QUESTIONS

1. Which of the following statements is false?
(a) If $A = \{x : x$ is a letter in the word 'life'$\}$ and
$B = \{x : x$ is a letter in the word 'file'$\}$ then
$A = B$
(b) The set $\{1, \{2, 3\}\}$ has four subsets
(c) $P(\phi) = \{\phi, \{\phi\}\}$
(d) For any two sets A and B,
$(A - B) \cap (B - A) = \phi$

2. A, B, C are arbitrary sets. Which of the following statements is true?
(a) $A \cap B = \phi, B \cap C = \phi \Rightarrow A \cap C = \phi$
(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
(c) If A has n elements and B has m elements, $n \neq m$ then $A - B$ has $n - m$ elements
(d) $(A - B) \cup (B - A) = A \cup B$

3. Match the sets in I with those in II.

I	II
(A) $A \times (B - C)$	(i) ϕ
(B) $A - (B \cup C)$	(ii) $(A \cap C) \times B$
(C) $(A - B) \cap (B - A)$	(iii) $(A \times B) - (A \times C)$
(D) $(A \times B) \cap (C \times B)$	(iv) $(A - B) - C$

Then the correct matching is
(a) (A) – (iii) (B) – (i) (C) – (iv) (D) – (ii)
(b) (A) – (iii) (B) – (iv) (C) – (i) (D) – (ii)
(c) (A) – (iii) (B) – (i) (C) – (ii) (D) – (iv)
(d) (A) – (ii) (B) – (iii) (C) – (iv) (D) – (i)

4. (i) If $A = \{a, b, \{a, c\}, \phi\}$ then $\{a\} - \{A\} = \phi$
(ii) $\bar{A} - \bar{B} = B - A$
Of the statements (i) and (ii),
(a) Both are true
(b) Both are false
(c) (i) is true and (ii) is false
(d) (i) is false and (ii) is true

5. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{2, 4, 6\}$, $C = \{1, 3, 5, 7\}$, $D = \{2, 3, 6\}$ and $E = \{2, 6\}$. Which set can equal X if we are given the information $X \subseteq A$ but $X \not\subseteq B$?
(a) X = B, E
(b) X = A, C, D
(c) X = C only
(d) none of the sets can equal X.

6. Let the universal set E be the set of all natural numbers less than 20. Let
$A = \{x \mid x \text{ is a prime less than } 23\}$
$B = \{x \mid x \text{ is divisible by } 3, x \leq 15\}$ and
$C = \{x \mid x \text{ is an odd number less than } 18\}$
Then $(A \cap B) \cup (A \cap C) =$
(a) $\{3, 5, 7, 11, 13\}$
(b) $\{3, 5, 7, 11, 17\}$
(c) $\{5, 7, 11, 13, 17\}$
(d) $\{3, 5, 7, 11, 13, 17\}$

7. It is known that in a group of people each of whom speaks atleast one of the languages English, Hindi and Bengali. 31 speak English, 36 speak Hindi, 27 speak Bengali, 10 speak both English and Hindi, 9 speak both English and Bengali, 11 speak both Hindi and Bengali. Then the minimum number of people in the group is
(a) 60 (b) 55
(c) 64 (d) None of these

8. In a survey of 1000 families in a township it was found that 458 families had T.V. sets, 820 had Radiogram, 600 had Tape Recorders; 294 families had both T.V. set and Radiogram, 277 had both Radiogram and Tape Recorder, 190 had both Tape Recorder and T.V. set. Then the number of families having Radiogram only is
(a) 132 (b) 123
(c) 83 (d) None of these

9. Out of 100 students in a college, 39 play tennis, 58 play cricket and 32 play hockey, 10 play cricket and hockey, 11 play hockey and tennis, 13 play tennis and cricket. The number of students playing tennis and cricket but not hockey is
(a) 8 (b) 5
(c) 24 (d) None of these

10. Which of the following statements are true?
(a) $P(A) \cap P(B) = P(A \cap B)$
(b) $A \backslash (B \backslash C) = (A \backslash B) \backslash C$
(c) $n(A) = n(B) \Rightarrow A = B$
(d) $n(A \backslash B) = n(A) - n(B)$

11. In a room containing 28 people there are 18 people who speak English, 15 people who speak Hindi and 22 people who speak Kannada, 9 persons speak both English and Hindi, 11 persons speak both Hindi and Kannada, whereas 13 people speak both Kannada and English. Then the number of people who speak all the three languages is
(a) 9 (b) 8
(c) 7 (d) 6

12. The results of an examination (having three parts) are as follows:-

Failed in all the three parts	20
Failed in Part I only	50
Failed in Part II only	45
Failed in Part III only	40
Failed in Parts I and II	30
Failed in Parts II and III	33
Failed in Parts III and I	23

Then the total number of candidates who failed is
(a) 181 (b) 135
(c) 46 (d) None of these

KEY

1. (c)	**2.** (b)	**3.** (b)	**4.** (d)
5. (b)	**6.** (d)	**7.** (c)	**8.** (a)
9. (a)	**10.** (a)	**11.** (d)	**12.** (a)

EXERCISE

1. If A = {1, 2, 3, 4}, B = {3, 4, 6, 8} and C = {2, 4, 5, 6} find $A \cup C$, $A \cap B$, $C \cap A$, $A \cup (B \cap C)$, $A - B$, $(C - A) - B$ and $C \Delta B$.

2. In a town 5600 persons read newspapers. Of these 3000 persons read The Hindu, 2000 read The New Indian Express and 300 read both. How many read neither?

3. If A = {1, 4}, B = {4, 5}, C = {5, 7} find $(A \times B) \cup (A \times C)$ and $(A \times B) \cap (A \times C)$.

4. Find x and y given that $(x^2 - 6, 2y + 1) = (-x, y - 1)$.

5. Find the number of elements in each of the following sets.
(a) P({a, b, {a, b}})
(b) $P(P\{\phi\})$
(c) $P(P(\phi))$

6. Let A = {a, b, d}, B = {d, b}. Find a suitable set X for each of the following:
(i) $X \subseteq A$ and $B \subseteq X$ **(ii)** $X \subseteq B$
(iii) $X \subseteq A$ and $X \subseteq B$ **(iv)** $A \subseteq X$ and $X \subseteq A$
(v) $X \not\subseteq A$ and $X \not\subseteq B$.

ANSWERS

1. $A \cup C = \{1, 2, 3, 4, 5, 6\}$; $A \cap B = \{3, 4\}$; $C \cap A = \{2, 4\}$
$A \cup (B \cap C) = \{1, 2, 3, 4, 6\}$; $A - B = \{1, 2\}$; $C - (A - B) = \{5\}$, $C \Delta B = \{2, 3, 5, 8\}$.

2. 900

3. $(A \times B) \cup (A \times C) = \{(1, 4), (1, 5), (4, 4), (4, 5), (1, 7), (4, 7)\}$.
$(A \times B) \cap (A \times C) = \{(1, 5), (4, 5)\}$.

4. $x = -3, 2$; $y = -2$

5. **(a)** 8, **(b)** 4, **(c)** 2

6. **(i)** X = B **(ii)** X = {b}, {d}, B, ϕ
(iii) X = B, ϕ **(iv)** X = A
(v) X = {a, c}.

3.2 RELATIONS

Introduction

1. Binary Relation: A and B are any two sets. A **binary relation** R from A to B is a subset of $A \times B$, $(a, b) \in R$ is also written as a R b. If R is a subset of $A \times A$, then we say that R is a (binary) relation on A.

If A has m elements and B has n elements then there are mn elements in $A \times B$ and so 2^{mn} subsets for $A \times B$. That is, there are 2^{mn} different relations from A to B.

The **domain of a relation** R is the set of all first elements of the ordered pairs which belongs to R and the **range** of R is the set of all second elements.

Example: Let A = {1, 2, 3, 4}, B = {3, 5}. Then
$A \times B = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5), (4, 3), (4, 5)\}$.
Consider R = {(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)}.

Then $R \subseteq A \times B$ and so R is a relation from A to B. The element (1, 3) of R is such that its first coordinate 1 is less than its second coordinate 3. Similarly, for other elements of R. Actually R is the relation "is less than".

The domain of R = {1, 2, 3, 4} = A.

The range of R = {3, 5} = B.

2. Representation of Relations on finite sets

1. Arrow diagram: Write the elements of A and B in two disjoint discs. Draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b. The picture thus obtained is known as the **arrow diagram**. The arrow diagram of the relation of the above example is given in **Figure** below.

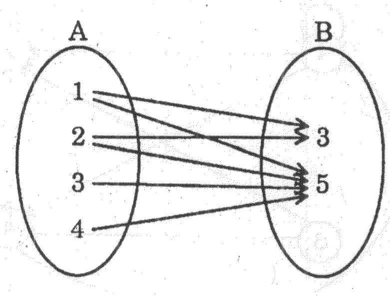


Fig.

2. Matrix of the relation: Form a rectangular array whose rows are marked by the elements of A and whose columns are marked by the elements of B. Put 1 or 0 in each position of the array according as a ∈ A is or is not related to b ∈ B. This array is known as the **matrix of the relation**.
In the example cited, 1 ∈ A is related to 3 ∈ B under R. Therefore, the entry in the first row, column under 3 of the matrix is 1. Similarly, fill up the other entries.

Note that (3, 3) ∉ R. So the entry in the 3rd row, first column is 0. Thus we get the following table.

	3	5
1	1	1
2	1	1
3	0	5
4	0	5

3. Directed graph of a relation: Let R be a relation from a finite set A to itself. Write the elements of A and encircle each element; then draw an arrow from each element a to each element b whenever a is related to b under R. The diagram so obtained is called the **directed graph of the relation R.**

Example: Let A = {1, 2, 3, 4} and let R = {(1, 1), (1, 2), (1, 4), (2, 4), (3, 2), (3, 4), (4, 1)} ⊆ A × A.

Then the directed graph of R is as in **Figure (a)**. Observe that there is an arrow from 1 to 1 since (1, 1) ∈ R.

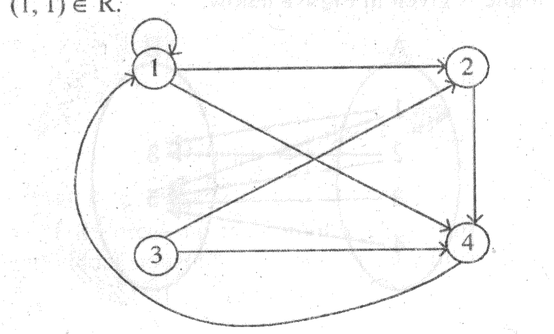


Fig. (a)

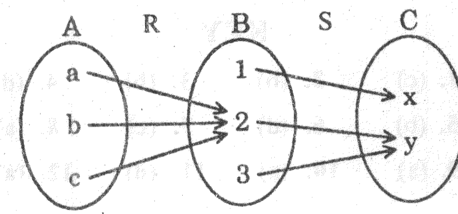


Fig. (b)

4. Inverse Relation: Let R be a relation from A to B. Then its **inverse relation**, denoted by R^{-1}, is a relation from B to A consisting of those ordered pairs which when reversed belong to R. That is,

R^{-1} = {(b, a) | (a, b) ∈ R}.

Example: Let A = {1, 2, 3} and B = {a, b, c} and R = {(1, a), (2, a), (3, c), (2, b)}. Then R^{-1} = {(a, 1), (a, 2), (c, 3), (b, 2)}.

Composition of Relations: Let A, B, C be sets. Suppose R is a relation from A to B and S is a relation from B to C. Then the relation from A to C, denoted by R ∘ S and defined by

R ∘ S = {(a, c)|(a, b) ∈ R, (b, c) ∈ S for some b ∈ B} is called the **composition of R and S.** (Some prefer to denote this relation by S ∘ R instead of R ∘ S).

Example: Let A = {a, b, c}, B = {1, 2, 3}, C = {x, y} and let R = {(a, 2), (b, 2), (c, 2)} and S = {(1, x), (2, y), (3, y)}. **Figure (b)** shows the arrow diagram of R and S. Note that there is an arrow from a to 2, followed by an arrow from 2 to y. Thus the element 2 ∈ B connects the element a ∈ A to the element y ∈ C. So (a, y) ∈ R ∘ S. No element of A is connected to the element x of C. Thus R ∘ S = {(a, y), (b, y), (c, y)}.

Types of Relations: Let R be a relation on a given set A.

Reflexive relation: R is said to be a **reflexive relation** if aRa for all a ∈ A; that is, (a, a) ∈ R for all a ∈ A.

R is not reflexive if there is at least one element a ∈ A such that (a, a) ∉ R. R is said to be **irreflexive** if no element is related to itself. i.e., (a, a) ∉ R. For any a in A.

Symmetric relation: R is said to be a **symmetric relation** if aRb $\Rightarrow$ bRa for all a, b $\in$ A; that is, (a, b) $\in$ R $\Rightarrow$ (b, a) $\in$ R for all a, b $\in$ A.

R is not symmetric if there exist a, b $\in$ A, a $\neq$ b such that (a, b) $\in$ R but (b, a) $\notin$ R.

Transitive relation: R is said to be a **transitive relation** if aRb, bRc $\Rightarrow$ aRc for all a, b, c $\in$ A; that is (a, b) $\in$ R, (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R for all a, b, c $\in$ A.

R is not transitive if there exist a, b, c $\in$ A, not necessarily distinct, such that (a, b) $\in$ R, (b, c) $\in$ R but (a, c) $\notin$ R.

Equivalence relation: R is said to be an **equivalence relation** if R is reflexive, symmetric and transitive.

Anti-symmetric relation: R is said to be an **anti-symmetric relation** if aRb, bRa $\Rightarrow$ a = b; that is (a, b) $\in$ R and (b, a) $\in$ R $\Rightarrow$ a = b.

R is not anti-symmetric if there exist elements a, b $\in$ A, a $\neq$ b such that (a, b) $\in$ R and (b, a) $\in$ R.

Partial Ordering Relation: R is said to be a **partial ordering** or a **partial order** if R is reflexive, anti-symmetric and transitive. A set A together with a partial ordering R is called a **partially ordered set** or **poset**.

Diagonal line of A × A: Δ = {(a, a)|a $\in$ A} is called the **diagonal line** of A × A or **equality** relation on A.

Example: Give an example of a relation defined on a suitable set which is

(a) an equivalence relation
(b) reflexive, symmetric but not transitive
(c) reflexive, transitive but not symmetric
(d) symmetric, transitive but not reflexive
(e) reflexive, but is neither symmetric nor transitive
(f) symmetric, but is neither reflexive nor transitive
(g) transitive, but is neither reflexive nor symmetric
(h) neither reflexive nor symmetric nor transitive
(i) a partial-ordering relation
(j) reflexive, transitive but not anti-symmetric

Solution: Let A = {1, 2, 3}

(a) R_1 = {(1, 1), (2, 2), (3, 3)} is reflexive, symmetric and transitive. So R_1 is an equivalence relation.

(b) R_2 = {(1, 1), (2, 2) (3, 3), (1, 2) (2, 1), (1, 3), (3, 1)} is clearly reflexive and symmetric. It is not transitive, for (2, 1) $\in R_2$, (1, 3) $\in R_2$ but (2, 3) $\notin R_2$.

(c) R_3 = {(1, 1), (2, 2), (3, 3), (1, 2)} is reflexive and transitive. It is not symmetric since (1, 2) $\in R_3$ but (2, 1) $\notin R_3$.

(d) R_4 = {(1, 1)} is symmetric and transitive but is not reflexive since (2, 2), (3, 3) are not in R_4.

(e) R_5 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)} is clearly reflexive. It is not symmetric since (1, 2) $\in R_5$ but (2, 1) $\notin R_5$. It is not transitive since (1, 2), (2, 3) $\in R_5$ but (1, 3) $\notin R_5$.

(f) R_6 = {(1, 1), (2, 2), (2, 3) (3, 2)} is clearly symmetric. It is not reflexive since (3, 3) $\notin R_6$. (3, 2), (2, 3) $\in R_6$ but (3, 3) $\notin R_6$. $\therefore$ R_6 is not transitive also.

(g) R_7 = {(1, 1), (2, 2), (1, 2)} is transitive. It is not reflexive since (3, 3) $\notin R_7$. It is not symmetric also for (1, 2) $\in R_7$ but (2, 1) $\notin R_7$.

(h) R_8 = {(1, 2) (2, 3)} is neither reflexive, nor symmetric, nor transitive.

(i) R_9 = {(1, 1), (2, 2) (3, 3)} is reflexive, anti-symmetric and transitive. So it is a partial ordering relation.

(j) R_{10} = A × A is reflexive and transitive but not anti-symmetric.
($\because$ (2, 3) $\in R_{10}$, (3, 2) $\in R_{10}$ and 2 $\neq$ 3).

Problem: Find the relation determined by the directed graph of the following **Figure**:

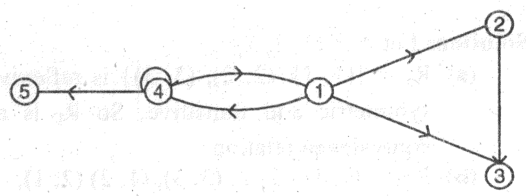


Fig.

Solution: Let $A=\{1,2,3,4,5\}$. The relation is

$$R=\{(1,2),(1,3),(1,4),(2,3),(4,1),(4,4),(4,5)\}$$

It is not reflexive $(\because (1,1)\notin R)$; not symmetric $(\because (1,2)\in R \text{ but } (2,1)\notin R)$; not transitive $(\because (1,4),(4,1)\in R \text{ but } (1,1)\notin R)$; not anti-symmetric $(\because (1,4),(4,1)\in R, 1\neq 4)$.

Problem: Let $X=\{1,2,3,......10\}$. Define ~ on X by $x\sim y \Leftrightarrow x+y=10$. Find if ~ is reflexive, symmetric or transitive.

Solution: $1+1=2\neq 10.\therefore 1 \not\sim 1$ and so ~ is not reflexive. Suppose $x\sim y$. Then $x+y=10 \therefore y+x=10$. That is, $y\sim x$. Thus ~ is symmetric.

Suppose $x\sim y, y\sim z$. Then $x+y=10$, $y+z=10$ so that $x=z$. So $x\not\sim z$ (taking $x=z=1, 1\not\sim 1$)

$\therefore$ ~ is not transitive.

Problem: Find the smallest and largest equivalence relations on a set A.

Solution: The largest equivalence relations on A is $A\times A$. $A\times A$ is called the **universal relation;** ϕ is called the **empty relation.** Let Δ be the diagonal line of $A\times A$. Then Δ is an equivalence relation on A. If R is any other equivalence relation on A, then R contains Δ. $\therefore \Delta$ is the smallest equivalence relation on A.

Problem: What is wrong with the following proof that symmetry and transitivity imply reflexivity?

$$a\sim b \Rightarrow b\sim a\,(\because \sim \text{is symmetric})$$

$$\therefore a\sim b, b\sim a \Rightarrow a\sim a\,(\because \sim \text{ is transitive})$$

$$\therefore a\sim a \text{ and so} \sim \text{ is reflexive}$$

Solution: Note that only for those a which are related to b by~ we get $a\sim a$ and not for all a.

$\therefore$ symmetry and transitivity $\not\Rightarrow$ reflexivity

Problem: Let $N=\{1,2,3,.....\}=$ set of all natural numbers. Verify if the relation ~ defined in N × N is an equivalence relation in each of the following cases:

(i) $(a, b)\sim(c, d) \Leftrightarrow a+d=b+c$

(ii) $(a,b)\sim(c,d)\Leftrightarrow ad=bc$

Solution: $N\times N=\{(a,b)|a,b\in N\}$

(i) $(a,b)\sim(c,d)\Leftrightarrow a+d=b+c$ (given)

(a) $(a,b)\sim(a,b)$ for all $(a, b)\in N\times N$, since $a+b=b+a$ is true. $\therefore$ ~ is reflexive.

(b) If $(a,b)\sim(c,d)$, then $a+d=b+c$; that is, $c+b=d+a$ and so $(c, d)\sim(a, b)$. Hence ~ is symmetric.

(c) If $(a,b)\sim(c,d)$ and $(c, d)\sim(e,f)$, then $a+d=b+c$ and $c+f=d+e$. Adding,
$a+d+c+f=b+c+d+e$
or $a+f=b+e$. $\therefore (a,b)\sim(e,f)$
Thus ~ is transitive.
Hence ~ is an equivalence relation.

(ii) $(a,b)\sim(c,d)\Leftrightarrow ad=bc$

(a) $(a,b)\sim(a,b)$ for all $(a,b)\in N\times N$ since $ab=ba$ is true. $\therefore$~ is reflexive.

(b) $(a,b)\sim(c,d) \Rightarrow ad=bc$
$\Rightarrow cd=da$
$\Rightarrow (c,d)\sim(a,b)$

$\therefore$ ~ is symmetric

(c) $(a,b)\sim(c,d)$ and $cb\sim(e,f)$
$\Rightarrow ad=bc$ and $cf=de$
$\Rightarrow adcf=bcde$
$\Rightarrow af=be$
$\Rightarrow (a,b)\sim(e,f)$

$\therefore$ ~ is transitive. So ~ is an equivalence relation.

Congruence: Let n be any positive integer. If a, b are any two integers and n divides a – b, then we say that " a is congruent to b modulo n" and we write $a\equiv b \pmod n$.

$$14\equiv 2(\text{mod } 4) \text{ since } 4 \text{ divides } 14-2.$$

Problem: Let n be any positive integer. Then the relation "is congruent to modulo n" is an equivalence relation on the set of all integers Z.

Solution: For any $a\in Z$, $a-a=0$ and $n\,|0$ (read as n divides 0)

$\therefore a\equiv a(\text{mod } n)$; so $\equiv$ is reflexive.

Now $a \equiv b \pmod{n}$ $\Rightarrow n \mid (a-b)$
$\Rightarrow n \mid (b-a)$
$\Rightarrow b \equiv a \pmod{n}$
$\therefore \equiv$ is symmetric.

Again, $a \equiv b \pmod{n}$, $b \equiv c \pmod{n}$
$\Rightarrow n \mid (a-b)$ and $n \mid (b-c)$
$\Rightarrow n \mid (a-b+b-c)$
$\Rightarrow n \mid (a-c)$
$\Rightarrow a \equiv c \pmod{n}$

$\therefore \equiv$ is transitive. Thus $\equiv$ is an equivalence relation.

Problem: Show that the relation of set inclusion $\subseteq$ is a partial order in P(S).

Solution: Let A, B, C $\in$ P(S).

(i) Since $A \subseteq A$, $\subseteq$ is reflexive

(ii) $A \subseteq B$, $B \subseteq A \Rightarrow A = B$. $\therefore \subseteq$ is anti-symmetric

(iii) $A \subseteq B$, $B \subseteq C \Rightarrow A \subseteq C$. $\therefore \subseteq$ is transitive.

Thus $\subseteq$ is a partial ordering relation in P(S).

Partition: Consider the set A = {1, 2, 3, 4, 5, 6} and the sets $A_1 = \{1, 2\}$, $A_2 = \{3, 5, 6\}$, $A_3 = \{4\}$.
Then the family $A = \{A_1, A_2, A_3\}$ has the following properties:

(i) A_1, A_2, A_3 are non-empty subsets of A.

(ii) $A = A_1 \cup A_2 \cup A_3$.

(iii) For any sets A_i, A_j either $A_i = A_j$ or $A_i \cap A_j = \phi$.

Such a family of sets is called a partition of A. Precisely, let A be any set. A collection of pairwise, disjoint, non-empty subsets of A whose union is A is called a **partition** of A. Also every equivalence relation on a set A gives rise to a partition of the set into **equivalence classes** or **equivalence sets**. Conversely, any partition of a set determines an equivalence relation R such that the members of the partition are precisely the equivalence classes defined by R.

Let R be an equivalence relation on A. For any $a \in A$, the set $[a] = \{x \in A \mid x \text{ R } a\}$ is called the **equivalence class** determined by a.

Problem: Find the partition of the set A = {1, 2, 3} defined by the equivalence relation aRb $\Leftrightarrow$ a = b.

Solution: R = {(1, 1), (2, 2), (3, 3)}. The members of the partition are precisely the equivalence classes determined by the elements 1, 2, 3 of A.

$[1] = \{x \in A \mid x \text{ R } 1\} = \{x \in A \mid x = 1\} = \{1\}$.
Similarly, [2] = {2} and [3] = {3}. Hence the partition is {{1}, {2}, {3}}.

Problem: Find the equivalence relation induced by the partition {{1}, {2, 3}} of the set {1, 2, 3}.

Solution: Let R be the equivalence relation induced by the given partition.

R is defined by

aRb $\Leftrightarrow$ a and b both belong to the same member of the partition.

Let $\{1\} = A_1$, $\{2, 3\} = A_2$.

Since, 1 and 1 belong to A_1, $(1, 1) \in R$.

Similarly, $(2, 2), (3, 3) \in R$.

Also 2 and 3 both belong to A_2.

$\therefore (2, 3), (3, 2) \in R$.

Thus R = {(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)} which is clearly an equivalence relation.

OBJECTIVE QUESTIONS

1. Which of the following statements is true?
   (A) Every equivalence relation is a partial-ordering relation.
   (B) The number of relations from A = {x, y, z} to B = {1, 2} is 64.
   (C) The empty relation ϕ is reflexive.
   (D) The properties of a relation being symmetric and being anti-symmetric are negatives of each other.
2. Which of the following statements is false?
   (A) If R is reflexive, then $R \cap R^{-1} \neq \phi$.
   (B) $R \cap R^{-1} = \phi \Rightarrow$ R is anti-symmetric.
   (C) If R, R′ are equivalence relations in a set A, then $R \cap R'$ is also an equivalence relation in A.
   (D) If R, R′ are reflexive relations in A, then R − R′ is reflexive.
3. Let A = {1, 2, 3, ... }. Define ~ by x ~ y $\Leftrightarrow$ x divides y. Then ~ is
   (A) reflexive, but not a partial-ordering
   (B) symmetric
   (C) an equivalence relation
   (D) a partial-ordering relation
4. Let A = {1, 2, 3}. Then the relation S = {(1, 1), (2, 2)} is

(A) symmetric only
(B) anti-symmetric only
(C) both symmetric and anti-symmetric
(D) an equivalence relation

5. Let A = {1, 2, 3, 4}. Let ~ = {(1, 2), (1, 3), (4, 2)}. Then ~ is
(A) not anti-symmetric
(B) transitive
(C) reflexive
(D) symmetric

6. Let R = {(1, 2), (2, 3), (3, 3)} be a relation defined on A = {1, 2, 3}. Then R ∘ R (= R^2) is
(A) R itself
(B) {(1, 2), (1, 3), (3, 3)}
(C) {(1, 3), (2, 3), (3, 3)}
(D) {(2, 1), (1, 3), (2, 3)}

7. The universal relation A × A on A is
(A) an equivalence relation
(B) anti-symmetric
(C) a partial ordering relation
(D) not symmetric and not anti-symmetric

8. The total number of different partitions of a set having four elements is
(A) 16 (B) 8
(C) 15 (D) 4

9. A partition of {1, 2, 3, 4, 5} is the family
(A) {{1, 2}, {3, 4}, {3, 5}}
(B) {ϕ, {1, 2}, {3, 4}, {5}}
(C) {{1, 2, 3}, {5}}
(D) {{1, 2}, {3, 4, 5}}

KEY

1. (B)	2. (D)	3. (D)	4. (C)
5. (B)	6. (C)	7. (A)	8. (C)
9. (D)			

EXERCISE

1. In the set Q of all rationals, let ~ be defined by $a \sim b \Leftrightarrow a = \frac{1}{b}$. Is Q (i) an equivalence relation (ii) a partial ordering relation?

2. In N = {1, 2, 3, ...} define a relation ρ by aρb ⇔ a is a multiple of b. Is ρ a partial order in N?

3. Find all the relations from {1, 2} to {3, 5}.

4. Express the relation ~ from A = {3, 5, 7, 9, 11} into B = {2, 6, 8, 10} as a set of ordered pairs defined by the sentences **(i)** ~: "is one less than" **(ii)** ~: "is a multiple of" **(iii)** a ~ b ⇔ a and b are relatively prime (Two integers are said to be relatively prime to each other, if their greatest common divisor is 1. For example, 4 and 15 are relatively prime) **(iv)** ~: "divides".

5. For each of the following relations on the set {1, 2, 3, 4}, decide whether it is reflexive, symmetric, anti-symmetric, transitive.
(a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
(b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
(c) {(2, 4), (4, 2)}
(d) {(1, 2), (2, 3), (3, 4)}
(e) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

6. Let R be a reflexive relation on a set A. Show that R^n is reflexive for all positive integers n.
$$\left(R^n = \underbrace{R \circ R \circ \ldots\ldots \circ R}_{n \text{ times}}\right).$$

7. What can you say about the equality relation Δ and any reflexive relation R on a set A?

8. How many relations are there on a set with n elements which are reflexive and symmetric?

9. On B = {1, 2, 3}, find which of the following relations is a partial-ordering relation?
(a) R_1 = {(1, 2), (3, 2), (2, 2), (2, 3)}
(b) R_2 = {(3, 2)}
(c) R_3 = {(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)}
(d) R_4 = {(1, 1)}.

10. Verify if the set {1, 2, 3, 4} with the usual ≤ is a poset.

11. If a relation R is a partial order in a set S, show that the inverse relation R^{-1} is also so.

12. Find the partition of the set of all integers in which the equivalence relation a ~ b ⇔ a − b is a multiple of 5 is defined.

13. Find the equivalence relations induced by the following partitions in the set A = {a, b, c, d}:-
   (i) {{a}, {b}, {c}, {d}}
   (ii) {{a, c}, {b, d}}
   (iii) {{a, c, d}, {b}}
   (iv) {{a, b, c, d}}

ANSWERS

1. **(i)** ~ is not an equivalence relation since it is not reflexive. **(ii)** ~ is not a partial ordering relation for the same reason.
2. Yes.
3. Members of P(S), where S = {(1, 3), (1, 5), (2, 3), (2, 5)}
4. **(i)** {(5, 6), (7, 8), (9, 10)}
   (ii) ϕ
   (iii) {(3, 2), (3, 8), (3, 10), (5, 2), (5, 6), (5, 8), (7, 2), (7, 6), (7, 8), (7, 10), (9, 2), (9, 8), (9, 10), (11, 2), (11, 6), (11, 8), (11, 10)}
   (iv) {(3, 6), (5, 10)}
5. **(a)** not reflexive, not symmetric, not anti-symmetric, transitive only
   (b) reflexive, symmetric, transitive
   (c) not reflexive, symmetric, not transitive, not anti-symmetric
   (d) not reflexive, not symmetric, not transitive, only anti-symmetric
   (e) not reflexive, not symmetric, not transitive, not anti-symmetric
6. Hint: Use induction
7. $\Delta \subseteq R$
8. $2^{\frac{n(n-1)}{2}}$
9. All the relations are not partial orders.
10. Yes.
11. Hint: Use the definition of R^{-1}.
12. {[0], [1], [2], [3], [4]}
13. **(i)** R = {(a, a), (b, b), (c, c), (d, d)}
   (ii) R = {(a, a), (b, b), (c, c), (d, d), (a, c), (c, a), (b, d), (d, b)}
   (iii) R = {(a, a), (b, b), (c, c), (d, d), (a, c), (c, a), (a, d), (d, a), (c, d), (d, c)}
   (iv) R = A × A

3.3 FUNCTIONS

1. Function: Let A and B be non-empty sets. A function or a map f from A into B is a rule which assigns to **each** element a ∈ A, a **unique** element b ∈ B. We write b = f(a). The element b is called the **image of a** under f; a is called the **pre-image of b.**

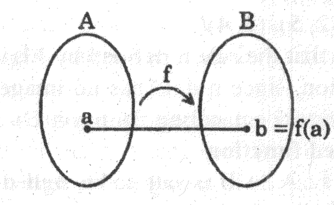


Fig.

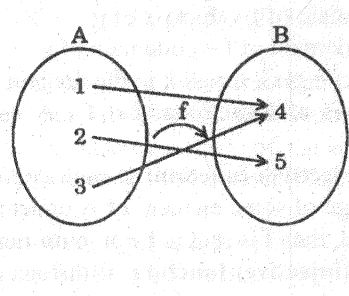


Fig.

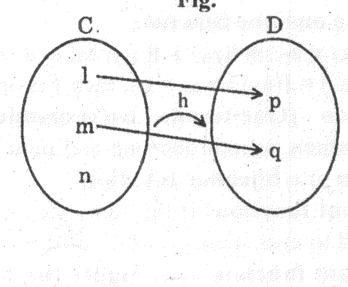


Fig. (a)

We write f : A → B or $A \xrightarrow{f} B$ (read as f is a function from A into B).

The set A is called the domain of f; the set B is called the codomain of f. The set f(A) = {f(a)/a ∈ A} is called the range of f.

Graph of a function: Let f : A → B be a function. Then for each a ∈ A, we get the unique ele-

ment f(a) in B. Thus, for each $a \in A$, we have an ordered pair (a, f(a)). The collection of all such ordered pairs (a, f(a)) where $a \in A$, constitutes a subset of $A \times B$. This subset is called the **graph of f** and is denoted by **graph f**.

Thus, graph $f = \{(a, f(a)) | a \in A\}$

For example, the graph of f defined by the adjoining diagram is

$\{(1, 4), (2, 5), (3, 4)\}$

Observe that the rule h defined by **Figure (a)** is not a function, since $n \in C$ has no image (that is, there is no arrow emanating from $n \in C$).

Well-defined function

A function $f : A \to B$ is said to be **well-defined** if $a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$, $a_1, a_2 \in A$.

Equal functions: Two functions f and g are said to be **equal** (written as f = g) if

(i) domain of f = domain of g

(ii) codomain of f = codomain of g

(iii) f(x) = g(x), for all x in the domain of f.

2. Types of Functions: Let $f : A \to B$ be a function.

Onto (surjective) function: If each element of B is the image of some element of A under f, that is, if f(A) = B, then f is said to be an **onto** function.

One-one (injective) function: If distinct elements in A have distinct images in B under f, then f is said to be a **one-one function**.

i.e., if $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$, $a_1, a_2 \in A$

or if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$, then f is one-one.

Bijection (One-to-one correspondence) A function which is both one-one and onto is called a **bijection** or a **bijective function**.

Constant function: If the same element $b \in B$ is assigned to each element of A, then the function is a **constant function**. (see **Figure (b)**, for example)

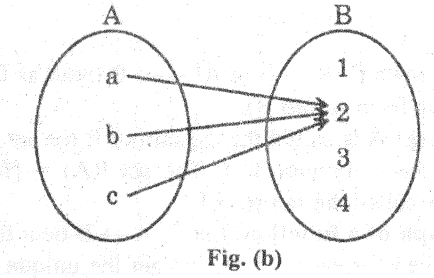


Fig. (b)

Identity function: Let A be any set. Then the function $f : A \to A$ defined by f(a) = a, for all $a \in A$ is called the **identity** function, denoted by 1_A. (see **Figure (c)**, for example)

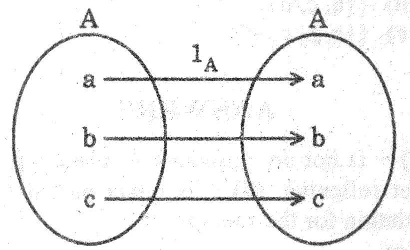


Fig. (c)

The identity function on any set is always a bijection.

Absolute value function: R – set of all real numbers. Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Then f is called the **absolute value function:** we write $f(x) = |x|$.

The absolute value function is not one-one since 2, –2 have the same image 2; is not onto also, since –1 is not the image of any element of R.

Characteristic Function: Let A be any set. For each subset S of A define a function

$\chi_S : A \to \{0, 1\}$ by

$$\chi_S(a) = \begin{cases} 1, & \text{if } a \in S, \\ 0, & \text{if } a \in A - S \end{cases}$$

χ_S is called the **Characteristic function** of S. Note that χ_A is a constant function, since it sends all the elements of A to 1.

Restriction and Extension of a function

Let $f : A \to B$ and $S \subseteq A$. Then the function $g : S \to B$ defined by g(s) = f(s) for all $s \in S$ is called the **restriction of f to S** and is denoted by $f | S$. The function f is called an **extension of g**.

Let $f : R \to R$ such that $f(x) = 2x$. Then $f \mid N = \{(1, 2), (2, 4), (3, 6), (4, 8), \ldots\}$ where, $N = \{1, 2, 3, \ldots\}$.

The function $F = \{(1, 2), (3, 2), (5, 6), (7, 8)\}$ is an extension of $\{(3, 2), (5, 6)\}$.

Let R^+ = set of all positive real numbers and $f(x) = |x|$, $x \in R$. Then $g : R^+ \to R$, such that $g(x) = x$ is the restriction of f on R^+ and so $g = f \mid R^+$.

Successor function: $N = \{1, 2, 3, \ldots\}$. The function $s : N \to N$ with $s(n) = n + 1 \;\; \forall\, n \in N$ is called (Peano's) **Successor function.** Note that s is not onto, since $1 \notin s(N)$; but s is one-one.

Boolean function: Let $B = \{0, 1\}$ and let $B^n = \{(x_1, x_2, \ldots x_n) \mid x_i = 0 \text{ or } 1\}$. Then a function $f : B^n \to B$ is called a **Boolean function.**

Let x, y be n-bit sequences in B^n. Let $f(x, y) = x \wedge y$.

$$\text{Then } x \wedge y = \begin{cases} 1, & \text{if } x = 1 \text{ and } y = 1 \\ 0, & \text{otherwise} \end{cases}$$

Note that this is nothing but an AND gate.

Hash function: Consider 11 boxes numbered 0, 1, 2, ...10. We want to store 3 values (integers) in these boxes. Hash function helps us to choose a box to store a particular number. For example, an integer n may be stored in box $h(n) = n \pmod{11}$. 12 is stored in box 1, 16 is stored in box 5, 39 is stored in box 6, 63 is stored in box 8 ($12 \equiv 1 \pmod{11}$, $16 \equiv 5 \pmod{11}$, $39 \equiv 6 \pmod{11}$ and $63 \equiv 8 \pmod{11}$). When we want to store 27 (natural choice is box 5 since $27 \equiv 5 \pmod{11}$), a **collision** occurs as box 5 is occupied by 16. To resolve this difficulty choose the next highest numbered empty box, namely box 7 and store 27. This policy is known as **collision resolution policy.** (see **Figure** below)

Pigeonhole principle: If n pigeonholes are occupied by n + 1 or more pigeons, then atleast one pigeonhole is occupied by more than one pigeon. In other words, if f is a function with finite domain D and target T where $|D| > |T|$, then f is not one-one (Here $|D|$ stands for the number of elements in D).

Composite of functions: Let $f : A \to B$ and $g : B \to C$. Then the **composite** of f and g, denoted by $g \circ f$, is a function from A into C defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & A \arrow[d, "g"] \\ & C \end{tikzcd}$$

Fig.

For example, if $f : R \to R$ and $g : R \to R$ are defined by $f(x) = x + 2$, $g(x) = x - 2$

then $(g \circ f)(x) = g(f(x)) = g(x + 2) = x + 2 - 2 = x$ and

$(f \circ g)(x) = f(g(x)) = f(x - 2) = x - 2 + 2 = x.$

Also $(f \circ f)(x) = f(f(x)) = f(x + 2) = x + 2 + 2 = x + 4$

$(g \circ g)(x) = g(g(x)) = g(x - 2) = x - 2 - 2 = x - 4$

Inverse Function: Let $f : A \to B$ be a bijection. Then the map $f^{-1} : B \to A$ which associates with each element $b \in B$, a unique element $a \in A$ such that $f(a) = b$ is called an **inverse function** of f. Note that f^{-1} is also a bijection and $f(a) = b \Leftrightarrow f^{-1}(b) = a$

Also (i) $(f^{-1})^{-1} = f$

(ii) $f \circ f^{-1} = 1_B$, $f^{-1} \circ f = 1_A$

(iii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

	12				16	39	$\boxed{27}$	63		
0	1	2	3	4	5	6	7	8	9	10

Fig. Hashing with collision

Countable Sets: A set A is said to be **countable**, if it can be put in one-to-one correspondence with the set $\{1, 2, 3, \ldots\}$ of all natural numbers.

For example, consider the set Z of all integers and the set N of all natural numbers.

The one-to-one correspondence between Z and N is shown below.

N:	1	2	3	4	5	6	....
	↓	↓	↓	↓	↓	↓	
Z:	0	1	−1	2	−2	3	....

The function $f : N \to Z$ defining the above one-to-one correspondence is given by

$$f(n)=\begin{cases}\frac{1}{2}n, & \text{if } n \text{ is even}\\ \frac{1}{2}(1-n), & \text{if } n \text{ is odd}\end{cases}$$

An infinite set which is not countable is called an **uncountable** set.

Problems with Solutions

Problem: Prove that the function $f : R \to R$ defined by $f(x) = 3x - 1$, $x \in R$ (= set of all real numbers) is a bijection. Find also f^{-1}.

Solution: Clearly f is well-defined.

Now $f(x) = f(y) \Rightarrow 3x - 1 = 3y - 1$
$\Rightarrow x = y$

$\therefore$ f is one-one.

Let $y \in$ codomain R be arbitrary. To show that f is onto, we have to find an $x \in$ domain R such that $f(x) = y$.

If $f(x) = y$, then $3x - 1 = y$, so that $x = \frac{y+1}{3}$.

$\therefore$ for $y \in R$ there exists $x = \frac{y+1}{3} \in$ domain R such that

$$f(x) = 3x - 1 = \frac{3(y+1)}{3} - 1 = y \therefore f \text{ is onto.}$$

Thus f is a bijection.

$\therefore f^{-1} : R \to R$ exists and is defined by

$$f(x) = 3x - 1 = \frac{3(y+1)}{3} - 1$$

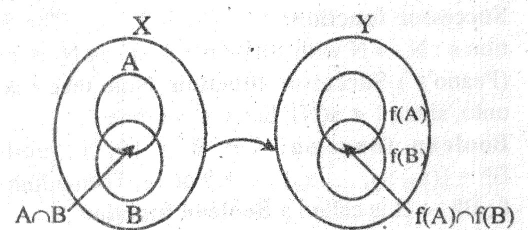
$$f^{-1}(x) = \frac{x+1}{3}.$$

Problem: A is any finite set. If $f : A \to A$ is one-one, show that f is onto.

Solution: Since $f : A \to A$ is a function, $f(A) \subseteq A$. Also f is one-one $\Rightarrow |f(A)| = |A|$. Thus $f(A) = A$. So f is onto.

Problem: Let $f : X \to Y$. Prove that f is one-one (written also as $1 - 1$) $\Leftrightarrow f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$.

Solution



$A \cap B \subseteq A \Rightarrow f(A \cap B) \subseteq f(A)$.

Similarly, $f(A \cap B) \subseteq f(B)$

$\therefore f(A \cap B) \subseteq f(A) \cap f(B)$ always. ... (1)

$\Rightarrow$ Suppose f is $1 - 1$.

In view of (1), it is enough to prove that $f(A) \cap f(B) \subseteq f(A \cap B)$

Now $y \in f(A) \cap f(B) \Rightarrow y \in f(A) \wedge y \in f(B)$
$\Rightarrow y = f(a), y = f(b)$ for some $a \in A, b \in B$
$\Rightarrow f(a) = f(b)$
$\Rightarrow a = b = x$ say ($\because$ f is $1 - 1$)
$\Rightarrow y = f(x), x \in A \cap B$
$\Rightarrow y \in f(A \cap B)$

Thus $f(A) \cap f(B) \subseteq f(A \cap B)$ and so
$f(A \cap B) = f(A) \cap f(B)$

Suppose $f(A \cap B) = f(A) \cap f(B)$. ... (2)

To prove f is $1 - 1$ let $f(x_1) = f(x_2) = t$ say. Assume that $x_1 \neq x_2$.

Put $A = \{x_1\}$, $B = \{x_2\}$. Then $A \cap B = \phi$ and so $f(A \cap B) = f(\phi) = \phi$.

Also $t = f(x_1) \in f(A)$, $t = f(x_2) \in f(B)$

$\therefore t \in f(A) \cap f(B) = f(A \cap B) = \phi$ (on using (2))

This is a contradiction and so $x_1 = x_2$

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

i.e., f is $1 - 1$.

Problem: Let $f : A \to B$ and $g : B \to C$. Prove or disprove:-

(a) $g \circ f$ is $1-1 \Rightarrow f$ is $1-1$

(b) $g \circ f$ is onto $\Rightarrow g$ is onto

(c) g is onto $\Rightarrow g \circ f$ is onto

(d) f is $1-1 \Rightarrow g \circ f$ is $1-1$

Solution (a): To prove $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$, $a_1, a_2 \in A$

Now $f(a_1) = f(a_2) \Rightarrow g(f(a_1)) = g(f(a_2))$ ($\because$ g is a function)

$\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2)$

$\Rightarrow a_1 = a_2$ ($\because g \circ f$ is $1-1$)

(b) To prove $g(B) = C$.

Since $g : B \to C$ is a function, $g(B) \subseteq C$.

$g \circ f : A \to C$ is onto $\Rightarrow (g \circ f)(A) = C$

$\Rightarrow g(f(A)) = C$

Since $f : A \to B$ is a function, $f(A) \subseteq B$

$\therefore g(f(A)) \subseteq g(B)$ ($\because$ g is a function)

Thus $C \subseteq g(B)$

$\therefore g(B) = C$.

(c) We disprove the statement.

Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{x, y\}$.

Let $f : A \to B$ and $g : B \to C$ be given by $f(a) = f(b) = f(c) = 2$; $g(1) = x$, $g(2) = g(3) = y$. clearly g is onto. Now $g \circ f : A \to C$ is such that

$(g \circ f)(a) = g(f(a)) = g(2) = y$

$(g \circ f)(b) = y$, $(g \circ f)(c) = y$

This means that x is not the image of any element of A and so $g \circ f$ is not onto.

(d) We disprove the statement.

Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{x, y\}$.

Let $f : A \to B$, $g : B \to C$ be such that $f(a) = 3$, $f(b) = 1$, $f(c) = 2$ and $g(1) = x$, $g(2) = y$, $g(3) = y$.

Clearly f is $1-1$.

But $(g \circ f) : A \to C$ is such that

$(g \circ f)(a) = g(3) = y$,

$(g \circ f)(c) = g(2) = y$ $\therefore (g \circ f)$ is not $1-1$.

Problem: Show that there is no one-to-one correspondence between a set A and its power set P(A).

Solution: Suppose there is a bijective function $f : A \to P(A)$. Then the matching is exactly in one of the following two ways:-

(i) an element is matched with a subset which contains that element.

(ii) an element is matched with a subset which does not contain it.

Call an element $a \in A$, a "dark" element if it is not in its matching subset; that is, if $a \notin f(a)$. Call an element $a \in A$, a "fair" element if it is in its matching subset; that is, if $a \in f(a)$.

Let T = set of all "dark" elements $= \{a | a \in A, a \notin f(a)\}$. Since $T \subseteq A$, $T \in P(A)$. Again, f is onto implies that there exists an element $a \in A$ such that $f(a) = T$.

If a is a "dark" element then $a \in T$ and hence $a \notin f(a) = T$, a contradiction.

If a is a "fair" element then $a \in f(a) = T$ and $a \in T$ implies $a \notin f(a) = T$, which is again a contradiction.

Thus, the assumption that there is a one-to-one correspondence between A and P(A) is false. $\therefore$ there is no one-to-one correspondence between A and P(A).

Problem: Show that the unit interval $I = [0, 1]$ is uncountable.

Solution: We assume that

(i) Every real number $x \in I$ can be represented as $x = 0.\, a_1\, a_2\, a_3 \ldots$ where a_i are integers, $0 \le a_i \le 9$. Conversely, every infinite decimal of the form $0.a_1a_2 \ldots$ represents some real number in I.

(ii) $0.999\ldots = 0.9 + 0.09 + 0.009 + \ldots$

$$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots$$

$$= \frac{9/10}{1 - \frac{1}{10}} = 1$$

(iii) If a number is rational, then its decimal expansion either terminates at a particular stage after which there are an infinite number of zeros or has cyclic repetition of the same digits for an infinite number of times.

(iv) If a number is irrational, then it can be represented as an infinite decimal.

Assume that $I = [0, 1]$ is countable. Then there is a one-to-one correspondence f between N (the set of all natural numbers) and I.

Let $f(1) = 0 \cdot a_{11}a_{12}a_{13} \ldots = r_1$ (say)

$f(2) = 0 \cdot a_{21}a_{22}a_{23} \ldots = r_2$ (say)

$f(3) = 0 \cdot a_{31}a_{32}a_{33} \ldots = r_3$ (say)

....

....

$f(n) = 0 \cdot a_{n1}a_{n2}a_{n3} \ldots a_{nn} \ldots = r_n$ (say).

Now define a real number $r = 0 \cdot a_1a_2a_3 \ldots a_n \ldots$, where for each n,

$$a_n = \begin{cases} 1, & \text{if } a_{nn} \neq 1 \\ 2, & \text{if } a_{nn} = 1 \end{cases}$$

Clearly, $r \in I$. Since each real number in I occurs among $r_1, r_2, \ldots$ it follows that $r = r_i$ for some i. But in r_i, the ith decimal digit is a_{ii} in r, the ith decimal digit is a_i and $a_i \neq a_{ii}$. This contradiction shows that I is not countable; that is, I is uncountable. This method of proof is known as **Cantor's diagonal procedure**. Since I is a subset of the set R of all real numbers, R is uncountable. (For, if R were countable, I being a subset of R must also be countable which is a contradiction).

OBJECTIVE QUESTIONS

1. The number of constant functions from {0, 1} to {2, 3} is
   (a) 1 (b) 2
   (c) 3 (d) 4
2. The pigeonhole principle asserts that if $f : A \to B$ and $|A| > |B|$ then
   (a) f is not onto
   (b) f is not 1 – 1
   (c) f is neither 1 – 1 nor onto
   (d) f can be 1 – 1
3. If f is the absolute value function defined by $f(x) = |x|$, then
   (a) f is a bijection (b) f is 1 – 1
   (c) f is onto (d) f is a function
4. Let $f : R \to R$ be defined by f(x) $= \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$ Then f (0.2333...) is
   (a) 1 (b) –1
   (c) is undefined (d) none of the above
5. B^A represents the set of all functions from A into B. If B = {x} then B^A has
   (a) 2 elements
   (b) only one element
   (c) more than one element
   (d) none of the above
6. Let $f : R \to R$, $g : R \to R$ such that $f(x) = \sin x$, $g(x) = x^2$. Then $(g \circ f)(x) =$
   (a) $\sin(x^2)$ (b) x^2
   (c) $\sin^2 x$ (d) $x^2 \sin x$
7. Let Z = set of all integers and let $f : Z \to Z$ be such that $f(x) = 3x$. Then
   (a) f has an inverse
   (b) f has no inverse
   (c) f is bijective
   (d) f is onto
8. Let A = {a, b, c, d}. If the function {(a, 1), (b, 1), (c, 1), (d, 0)} is a characteristic function of some subset S of A, then S =
   (a) {a, b, d} (b) {a, c, d}
   (c) {a, b, c} (d) {b, c, d}
9. Let Z = set of all integers. Which of the following functions $f : Z \to Z$ is onto?
   (a) $f(n) = n^2 + 1$ (b) $f(n) = n^3$
   (c) $f(n) = |n|$ (d) $f(n) = n - 1$
10. Which of the following is a function $f : R \to R$?
    (a) $f(x) = \frac{1}{x}$
    (b) $f(x) = \sqrt{x}$
    (c) $f(x) = \pm\sqrt{x^2 + 1}$
    (d) $f(x) = 2x^2 + 3x - 1$

ANSWERS

1. (b)	2. (b)	3. (d)	4. (a)
5. (b)	6. (c)	7. (b)	8. (c)
9. (d)	10. (d)		

EXERCISE

1. Let $f : A \to B$ and $g : B \to C$ be bijections. Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
2. If $f : A \to B$ is bijective, show that $f^{-1} : B \to A$ is also a bijection.
3. Express the function $f : R \to R$, $f(x) = 2x - 5$ as the composite of two functions.

4. Prove or disprove:

(i) Composition of functions is associative; that is, $h \circ (g \circ f) = (h \circ g) \circ f$, where $f : A \to B, g : B \to C, h : C \to D$.

(ii) Composition of functions is commutative; that is, $g \circ f = f \circ g$, where $f : A \to B$, $g : B \to C$.

5. Let $A = \{1, 2, 3, 4\}$. Find if the following subsets of $A \times A$ are graphs of functions from A into A:

(i) $\{(1,4),(4,1),(1,2),(4,3)\}$

(ii) $\{(4,1),(1,2),(2,3),(3,4)\}$

(iii) $\{(1,4),(2,1),(3,2),(4,3)\}$

(iv) $\{(4,1),(1,4),(2,1),(3,3)\}$

6. Show that the function $f : R \to R$ defined by $f(x) = \sin x$ is neither 1 – 1 nor onto.

7. Let S and T be subsets of set A. If χ_S is the characteristic function of S and if $\chi_S\chi_T$: $A \to \{0,1\}$ is defined by $\chi_S\chi_T(x) = \chi_S(x)\chi_T(x)$, show that $\chi_{S \cap T} = \chi_S\chi_T$.

ANSWERS

3. $f = g \circ h$ where $g(x) = 2x, h(x) = x - \frac{5}{2}$ for all $x \in R$.

4. (i) is true (ii) is false. Let $f : R \to R$, $g : R \to R$ be such that $f(x) = \sin x$, $g(x) = x^2$. Then $(g \circ f)(x) = \sin^2 x$ and $(f \circ g)(x) = \sin (x^2)$ so that $g \circ f \neq f \circ g$.

5. (ii), (iii), (iv) are graphs of functions.

3.4 ALGEBRA (Groups, Rings, Fields)

1. Introduction

Binary operation: Let A be a non-empty set. A **binary operation** on A is a function $*$ from $A \times A$ into A. The image of an ordered pair $(a,b) \in A \times A$ is written as $a * b$ (instead of $*(a, b)$). The set A together with an operation $*$ is denoted by $(A,*)$. We say that A is **closed under** $*$ if $a * b \in A$ for all $a, b \in A$.

The operations $\cup, \cap, \Delta$ and $-$ are all binary operations on P(A). The set $N = \{1, 2, \ldots\ldots\}$ is not closed under subtraction since $1 - 3 = -2 \notin N$; but N is closed under usual addition and multiplication.

Properties of operations: Let $*$ be a binary operation on a non-empty set A. Then,

(i) $*$ **is commutative,** if $a * b = b * a$ for all $a, b \in A$.

(ii) $*$ **is associative** if $a*(b*c) = (a*b)*c$ for all $a, b, c \in A$.

(iii) An element e in A is called an **identity element** for $*$ if $a * e = e * a = a$ for all $a \in A$. $e \in A$ is called a **left identity** or **right identity** for $*$ according as $e * a = a$ or $a * e = a$ for all $a \in A$.

(iv) Let e be the identity element for $*$. The **inverse** of an element a in A is an element $b \in A$ such that $a * b = b * a = e$. An element b in A is said to be a **left inverse** of a in A if $b * a = e$; b is called a **right inverse** of a if $a * b = e$.

(v) If A is closed under $*$, then $(A, *)$ is called a **groupoid.**

Note:

(i) If b is the inverse of a, then a is the inverse of b.

(ii) The relation "is an inverse of" is a symmetric relation.

(iii) If the operation on A is denoted by $*$, then A is said to be written **multiplicatively,** and the inverse of a in A is written as a^{-1}. If the operation is denoted by +, then A is said to be written **additively.** In this case, the identity element is denoted by 0 (called **zero element**) and the inverse of a in A is denoted by – a.

Example: Consider the set Z of all integers under + (usual addition). Let a, b, c, $\in$ Z be arbitrary. Then, Z is closed under + since $a + b \in Z$.

+ is associative, since $a + (b + c) = (a + b) + c$.

+ is commutative, since $a + b = b + a$.

Identity element for + is 0 since $a + 0 = 0 + a = a$.

Inverse of a is – a since $a + (-a) = (-a) + a = 0$.

Example: Consider the power set P(S) of a finite set S. Let $A, B \in P(S)$ be arbitrary; let $* = \cap$.

Since $A \cap B \in P(S)$, closure property holds. Clearly $\cap$ is commutative and associative. S is the identity element for $\cap$ since $A \cap S = A = S \cap A$. There is no inverse for A since we cannot find B $\in P(S)$ satisfying $A \cap B = S = B \cap A$.

2. Semi groups

Let A be a non-empty set. Then A is said to be a **semi group** if

(i) A is closed under a binary operation $*$

(ii) $*$ is associative.

Example: Let N be the set of all positive integers. Then $(N, +)$ and $(N, \cdot)$ are semi groups where $+$ and $\cdot$ are usual addition and multiplication respectively.

Example: Let $N' = N \cup \{0\} = \{0, 1, 2, \ldots\}$. Define $*$ on N' by $x * y = \max\{x, y\}$. (= maximum of x, y). Clearly N' is closed under $*$.

Now let $x, y, z \in N'$.

Then, $x * (y * z) = x * \max\{y, z\}$

$= \max\{x, \max\{y, z\}\}$

$= \max\{x, y, z\}$ and

$(x * y) * z = \max\{x, y\} * z$

$= \max\{\max\{x, y\}, z\}$

$= \max\{x, y, z\}$ so that

$x * (y * z) = (x * y) * z$.

$\therefore *$ is associative. Thus $(N', *)$ is a semi group.

Example: Let S be a finite set; let F(S) = collection of all functions $f : S \to S$.

Let $* = \circ$ = composition of functions. Since $*$ is an associative binary operation, $(F(S), *)$ is a semi group.

Example: Let $A = \{a, b, c, d\}$. $*$ is a binary operation on A given by the following multiplication table

*	a	b	c	d
a	a	b	c	d
b	b	a	a	b
c	c	b	a	a
d	d	a	a	a

From the above table, $d * b = a$, $b * c = a$, $a * c = c$ and $d * a = d$.

Now $d * (b * c) = d * a = d$ and

$(d * b) * c = a * c = c$

$\therefore d * (b * c) \neq (d * b) * c$. So $*$ is not associative. Hence $(A, *)$ is not a semi group.

Let $(S, *)$ be a semi group. An element ℓ in S is said to be a **left zero** of S if $\ell * x = \ell$ for all x in S. Similarly we define a **right zero** of S. For example, let $S = \{a, b, c\}$ and $*$ is given by

*	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Then a is a left zero for S since $a * a = a$, $a * b = a$, $a * c = a$, i.e., $a * x = a$ for all x in S. Note that b and c are also left zeros of S.

On the other hand, for the semi group $N' = N \cup \{0\}$ with usual multiplication, $0 \times x = 0$ for all $x \in S$. So $0 \in S$ is a left zero for S.

Problem: Let $(S, *)$ be a semi group and $0 \in S$ be a left zero of S. Prove that for any element x, $x * 0$ is also a left zero.

Solution: Fix $x \in S$. Let $s \in S$ be arbitrary.

Then $(x * 0) * s = x * (0 * s)$ ($\because *$ is associative)

$= x * 0$ (0 is a left zero $\Rightarrow 0 * s = 0$)

Thus $(x * 0) * s = x * 0$ for all $s \in S$.

$\therefore x * 0$ is a left zero of S.

3. Monoid

A semi group $(A, *)$ is said to be a **monoid**, if $*$ has an identity.

An element a in a monoid $(M, *)$ is said to be an **idempotent element**, if $a * a = a$.

A subset T of a monoid $(M, *)$ is said to be a **submonoid of M**, if $(T, *)$ itself is a monoid.

The semi group $(P(s), \cap)$ is a monoid since it has the identity S for $\cap$. The semi group $(N, \cdot)$ is also a monoid with 1 as the identity; but the semi group $(N, +)$ is not a monoid since $+$ has no identity. Considering $(N', *)$, 0 is the identity for $*$ and so $(N', *)$ is a monoid. Each $A \in P(S)$ is an idempotent for the monoid $(P(S), \cap)$ [$\because A \cap A = A$] whereas, 1 is the only idempotent in the monoid $(N, \cdot)$.

Problem: If $(M, *)$ is a commutative monoid, prove that the set T of all idempotent elements of M is a sub-monoid of M.

Solution: Let $T = \{a \in M | a * a = a\}$

Since M is a monoid, it has identity element e.

Now $e \in T$ since $e * e = e$.

Thus $T \neq \phi$.

Let $a, b \in T$. Then $a * a = a$, $b * b = b$.

To prove $a * b \in T$

Now $(a * b) * (a * b) = (a * a) * (b * b)$

[$\because$ $*$ is both commutative and associative]

$= a * b$

Thus $a * b \in T$; that is, T is closed under $*$. Since $*$ is associative on M it is associative on the subset T of M.

e is the identity element for T also.

$\therefore$ $(T, *)$ is a sub-monoid of $(M, *)$

4. Group

A monoid $(G, *)$ is a **group**, if each a in G has an inverse a′ in G. Thus, a **group** is a non-empty set G together with a binary operation $*$ defined on G satisfying,

(i) $*$ is associative i.e., $a * (b * c) = (a * b) * c$ for all a, b, c in G

(ii) there exists an element e in G called an identity element such that $a * e = e * a = a$ for all a in G.

(iii) For each a in G there exists an element a′ in G called an inverse of a such that $a * a' = a' * a = e$.

A group $(G, *)$ is said to be **commutative** or **abelian**, if $*$ is commutative.

i.e., $a * b = b * a$ for all a, b in G.

A group $(G, *)$ is said to be a finite group or an infinite group according as the set G is finite or infinite.

For a group G, **(i)** identity element is unique **(ii)** for any $a \in G$, inverse of a is unique **(iii)** cancellation laws hold good. That is, for all $a, b, c \in G$,

$a * b = a * c \Rightarrow b = c$ (left cancellation law)

$b * a = c * a \Rightarrow b = c$ (right cancellation law)

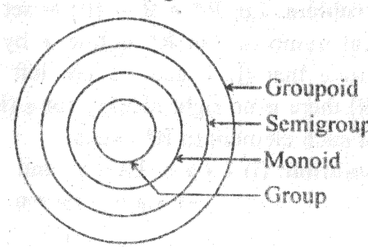


Fig.

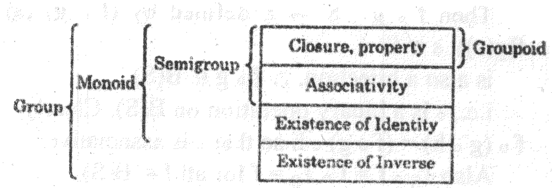


Fig.

Example: $(P(S), \Delta)$ is an abelian group for which ϕ is the identity element and the inverse of each element is itself ($\because$ $\phi \Delta A = A \Delta \phi = A$; $A \Delta A = \phi$).

The set of all integers, the set of all rationals, the set of all reals and the set of all complex numbers are abelian groups under usual addition. These are respectively known as the **additive group of integers**, the **additive group of rationals**, the **additive group of reals** and the **additive group of complex numbers.**

Problem: Show that a semi group with more than one idempotent element cannot be a group.

Solution: Let $(S, *)$ be a semi group with two idempotent elements x, y. That is, $x \neq y$ and

$x * x = x$ ... (1)

$y * y = y$ ... (2)

Assume that $(S, *)$ is a group with identity e.

Then $x * e = x$ and $y * e = y$ ... (3)

From (1) and (3), $x * x = x * e$.

Since S is a group, by left cancellation law,

$x = e$.

Similarly, $y * y = y * e$ yields $y = e$.

Thus $x = e = y$ which contradicts $x \neq y$

$\therefore$ $(S, *)$ cannot be a group.

Problem: Prove that the set B(S) of all bijections of a non-empty set S to itself is a group under composition of functions.

Solution: $B(S) = \{f | f : S \to S$ is a bijection$\}$ B(S) is non-empty since the identity function $I_S \in B(S)$.

Let $f, g, h \in B(S)$.

Then $f \circ g : S \to S$ defined by $(f \circ g)(s) = f(g(s)), s \in S$
is also a bijection. $\therefore f \circ g \in B(S)$.
i.e., $\circ$ is a binary operation on $B(S)$. Clearly $f \circ (g \circ h) = (f \circ g) \circ h$ so that $\circ$ is associative.
Also $1_S \circ f = f \circ 1_S = f$ for all $f \in B(S)$
$\therefore 1_S$ is the identity element.

Since f is a bijection, $f^{-1} : S \to S$ is also a bijection. $\therefore f^{-1} \in B(S)$ and $f \circ f^{-1} = f^{-1} \circ f = 1_S$.
$\therefore f^{-1}$ is the inverse of f.

Thus $(B(S), \circ)$ is a group. This group is non-commutative, when S has more than two elements. Take $S = \{1, 2, 3\}$. Define $f : S \to S$, $g : S \to S$ by $f(1) = 1$, $f(2) = 3$, $f(3) = 2$ and $g(1) = 2$, $g(2) = 1$, $g(3) = 3$. Then $(f \circ g)(1) = 3$ and $(g \circ f)(1) = 2$ so that $f \circ g \neq g \circ f$.

Problem: Let $(G, *)$ be a group. Prove the following:

(i) $(a * b)^{-1} = b^{-1} * a^{-1}$ for all $a, b \in G$

(ii) For $a \in G$, $a = a^{-1} \Rightarrow (G, *)$ is an abelian group.

(iii) $(a^{-1})^{-1} = a$.

Solution

(i)
$$\begin{aligned}(a * b) * (b^{-1} * a^{-1}) &= a * (b * b^{-1}) * a^{-1} \quad (\because * \text{ is associative})\\ &= a * e * a^{-1}\\ &= a * a^{-1}\\ &= e\end{aligned}$$

and
$$\begin{aligned}(b^{-1} * a^{-1}) * (a * b) &= b^{-1} * (a^{-1} * a) * b \quad (\because \text{ is associative})\\ &= b^{-1} * e * b\\ &= b^{-1} * b\\ &= e\end{aligned}$$

Thus $(a * b) * (b^{-1} * a^{-1}) = (b^{-1} * a^{-1}) * (a * b) = e$ which is of the form $x * y = y * x = e$ so that $y = x^{-1}$ i.e., $b^{-1} * a^{-1} = (a * b)^{-1}$.

(ii) Let $a, b \in G$ with $a = a^{-1}$, $b = b^{-1}$.
Then $a * b = (a * b)^{-1} = b^{-1} * a^{-1} = b * a$
$\therefore (G, *)$ is abelian.

(ii) We know $a * a^{-1} = a^{-1} * a = e$ and
$(a^{-1}) * (a^{-1})^{-1} = (a^{-1})^{-1} * a^{-1} = e$
Hence $a^{-1} * a = (a^{-1}) * (a^{-1})^{-1}$.
By left cancellation,
$a = (a^{-1})^{-1}$.

Note: 1. It is cumbersome to keep using $*$ for the group operation. Hereafter we shall drop it and write simply ab instead of $a * b$.

2. Let a be any element of a group G. For any positive integer n, $\underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$ is denoted by a^n. Define $a^{-n} = (a^{-1})^n$ and $a^0 = e$, the identity element of G. Then a^n is defined for all integers n.

Problem: If G is a group in which $(ab)^n = a^n b^n$ for three consecutive integers n and for all $a, b \in G$ show that G must be abelian.

Solution: Given

$$(ab)^n = a^n b^n \quad \ldots (1)$$
$$(ab)^{n+1} = a^{n+1} b^{n+1} \quad \ldots (2)$$
$$(ab)^{n+2} = a^{n+2} b^{n+2} \quad \ldots (3)$$

To prove that ab = ba for all a, b in G.

Now $(ab)^{n+1} = a^{n+1} b^{n+1}$
$\Rightarrow (ab)^n (ab) = (a^n a)(b^n b)$
$\Rightarrow (a^n b^n)(ab) = (a^n a)(b^n b)$ (from (1))
$\Rightarrow a^n(b^n a)b = a^n(ab^n)b$ (by associativity)
$\Rightarrow b^n a = ab^n$ (by cancellation law) ... (4)

Again, $(ab)^{n+2} = a^{n+2} b^{n+2}$
$\Rightarrow (ab)^{n+1}(ab) = (a^{n+1}a)(b^{n+1}b)$
$\Rightarrow (a^{n+1}b^{n+1})(ab) = (a^{n+1}a)(b^{n+1}b)$ (from (2))
$\Rightarrow a^{n+1}(b^{n+1}a)b = a^{n+1}(ab^{n+1})b$
$\Rightarrow b^{n+1}a = ab^{n+1}$ (by cancellation law)
$\Rightarrow (b^n b)a = a(b^n b)$
$\Rightarrow b^n(ba) = (ab^n)b$
$\Rightarrow b^n(ba) = (b^n a)b$ (from (4))
$\Rightarrow b^n(ba) = b^n(ab)$
$\Rightarrow ba = ab$ (by cancellation law)

Thus G is abelian.

Problem: Let $R^* = R - \{0\}$ = set of all non-zero real numbers. On R^* define $*$ by $a * b = |a|\, b$. Prove that **(i)** 1 and -1 are left identities for $*$ **(ii)** there is no right identity for $*$ **(iii)** right inverse of each element in R^* exists.

Solution: **(i)** $1 * a = |1|a = a$ and
$-1 * a = |-1|a = a$ for all a in R^*

$\therefore$ 1 and -1 are both left identities for $*$.

(ii) For any $a \in R^*$,

$a * e = a \Rightarrow |a|\, e = a$

$$\Rightarrow e = \frac{a}{|a|}$$

$$\Rightarrow e = 1$$

$$e = -1 \begin{cases} \text{if } a > 0 \\ \text{if } a < 0 \end{cases}$$

Thus the value of e depends upon a. So there is no right identity for $*$.

(iii) Taking $e = 1$, $a * b = e = 1 \Rightarrow |a|\, b = 1$

$\therefore \frac{1}{a}$ is a right inverse of a if $a > 0$ and $-\frac{1}{a}$ is a right inverse of a if $a < 0$.

Problem: Let P [x] be the set of all polynomials in x with integer coefficients (For example, $2 - 3x + 5x^2 + 7x^3$ is one such polynomial). Show that P[x] is an abelian group under addition of polynomials.

Solution: Let $p(x), q(x) \in P[x]$.

Clearly, $p(x) + q(x) \in P[x]$

Also + is associative in P[x]. The **zero polynomial** 0 (that is, the polynomial with all coefficients 0) is the identity element. The inverse of p(x) is $-p(x)$. Also $p(x) + q(x) = q(x) + p(x)$. Thus (P[x], +) is an abelian group.

A subset H of a group $(G, *)$ is said to be a **subgroup** of G if (i) H is closed under $*$ (closure property) (ii) $(H, *)$ is a group.

Example: $G = \{1, -1, i, -i\}$, where $i^2 = -1$ is a group under multiplication of complex numbers. The subset $S = \{1, -1\}$ is a subgroup of G with 1 as identity element. Each element of S is its own inverse. But $H = \{1, i\}$ is not a subgroup of G for H is not closed under multiplication ($\because i \cdot i = i^2 = -1 \notin H$).

Example: For any integer n, define $nZ = \{nz | z \in Z\}$. Then nZ is a subgroup of (Z, +), Z being the set of all integers.

5. Rings and Fields

Ring: A non-empty set R together with two binary operations on it denoted by + and $\cdot$ is said to be a **ring**, if

(a) (R, +) is an abelian group

(b) $(R, \cdot)$ is a semi group (closure property and associative property with respect to $\cdot$).

(c) The distributive law 2 hold;

i.e., for all $a, b, c \in R$

$a \cdot (b + c) = a \cdot b + a \cdot c$ (left distributive law)

$(b + c) \cdot a = b \cdot a + c \cdot a$ (right distributive law)

A ring $(R, +, \cdot)$ is said to be a **commutative ring**, if the binary operation $\cdot$ is commutative.

That is, $a \cdot b = b \cdot a$ for all a, b in R.

Example: The set of all integers under addition and multiplication is a commutative ring. It is usually called the **ring of integers**.

Example: Let $M_2(Z)$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,\ b,\ c,\ d \text{ are integers} \right\}$$

Then $M_2(Z)$ is a ring under addition and multiplication of matrices.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

Then $AB = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\therefore$ $AB \neq BA$ and hence $M_2(Z)$ is a non-commutative ring.

Field: A non-empty set F with at least two elements together with two binary operations on it denoted by + and $\cdot$ is said to be a **field**, if

(i) (F, +) is an abelian group.

(ii) $(F - \{0\}, \cdot)$ is a commutative group (0 stands for the identity element for +).

(iii) multiplication is distributive over addition. That is, $a \cdot (b + c) = a \cdot b + a \cdot c$ for all a, b, c in F.

Example: The set of all real numbers under usual addition and multiplication is a field. But $(Z, +, \cdot)$ is not a field, where Z = set of all integers.

6. Homomorphism: A **homomorphism** f of a groupoid $(A, \cdot)$ into a groupoid $(B, *)$ is a function $f : A \to B$ such that

$f(a_1 \cdot a_2) = f(a_1) * f(a_2)$ for all $a_1, a_2 \in A$.

Thus, if A and B are groups, f is a group homomorphism or a homomorphism of groups, if

$f(a_1 \cdot a_2) = f(a_1) * f(a_2)$ for all $a_1, a_2 \in A$.

In the case of rings and fields since there are two binary operations, the function $f : (R, +, \cdot) \to (R', +', \cdot')$ is called a **ring homomorphism** if

(i) $f(r_1 + r_2) = f(r_1) +' f(r_2)$

(ii) $f(r_1 \cdot r_2) = f(r_1) \cdot' f(r_2)$ for all $r_1, r_2 \in R$.

Example: Let $(R^+, \cdot)$ = group of positive real numbers under multiplication.

$(R, +)$ = group of all real numbers under addition.

Define $f: R^+ \to R$ by $f(x) = \log_{10}x$.

To prove $f(x \cdot y) = f(x) + f(y)$ since the binary operations for R^+ and R are $\cdot$ and + respectively.

Now $f(x \cdot y) = \log_{10}(xy) = \log_{10}x + \log_{10}y$
$= f(x) + f(y)$

$\therefore$ f is a homomorphism.

7. Types of homomorphisms

Let f: A $\to$ B be a homomorphism of a groupoid $(A, \cdot)$ into a groupoid $(B, *)$. Then

(a) f is called a **monomorphism**, if f is 1 – 1.

(b) f is called an **epimorphism**, if f is onto. In this case B is called the **homomorphic image** of A.

(c) f is called an **isomorphism**, if f is a bijection.

(d) A homomorphism f from a groupoid A to itself is called an **endomorphism**.

(e) An endomorphism of a groupoid A which is also a bijection is called an **automorphism**. That is, f is called an automorphism if $f : X \to X$ is a bijective homomorphism.

Example: Let Z = set of all integers, 2Z = set of all even integers. Then (Z, +) and (2Z, +) are groups where + stands for usual addition.

Define $f : Z \to 2Z$ by $f(x) = 2x$, $x \in \mathcal{Z}$.

For $x, y \in Z$, $f(x + y) = 2(x + y) = 2x + 2y$
$= f(x) + f(y)$

$\therefore$ f is a homomorphism.

Clearly, f is a bijection. $\therefore$ f is an isomorphism.

Example: Let G be an abelian group. Define $T : G \to G$ by $T(x) = x^{-1}$, for all $x \in G$. Clearly, T is a bijection.

Also $T(xy) = (xy)^{-1} = y^{-1}x^{-1}$ (by problem 54(i))

$= x^{-1}y^{-1}$ ($\because$ G is abelian)

$= T(x)\,T(y)$

$\therefore$ T is a homomorphism.

Thus, T is an automorphism of G.

8. Cyclic group: Let G be a group. If there exists an element $a \in G$, such that $G = \{a^n | n$ is an integer$\}$. Then G is called a **cyclic group** (generated by the element a) and a is called a **generator of G**. We write G = <a>.

The additive group of integers is cyclic with 1 as generator.

OBJECTIVE QUESTIONS

1. Which of the following statements is false?

(A) Every monoid is a semi group.

(B) Let Z be the set of all integers and (Z, *), an algebraic system, where the operation * is defined by n * m = maximum {n, m}. Then (Z, *) is a monoid.

(C) Every submonoid of a finite monoid is finite.

(D) The groupoid {0, 1, 2, 3} with * defined by

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

has a 0 and an identity 1.

2. Which of the following statements is true?

(A) Every abelian group is cyclic.

(B) Every commutative ring is a field.

(C) Every element of the ring of all integers (under usual addition and multiplication) has a multiplicative inverse.

(D) Every cyclic group is abelian.

3. Consider the statements
   (i) $\{-2, -1, 0, 1, 2\}$ is not a subgroup of the additive group of integers.
   (ii) $Z_n = \{0, 1, 2, \ldots n-1\}$, where n is a positive integer is a group under $\oplus$ defined by $a \oplus b = r$, where $a + b = qn + r$, $0 \le r < n$ (For example, when $n = 4$, $2 \oplus 3 = 1$).

   Of the two statements,
   (A) Both (i) and (ii) are true
   (B) (i) is true and (ii) is false
   (C) (i) is false and (ii) is true
   (D) Both (i) and (ii) are false
4. The relation "is isomorphic to" is
   (A) a partial ordering relation among groups
   (B) a total order among groups
   (C) an equivalence relation among groups
   (D) is irreflexive, symmetric and transitive among groups

KEY

1. (B) **2.** (D) **3.** (A) **4.** (C)

EXERCISE

1. If G is a group such that $x^2 = e$ for all x in G show that G must be an abelian.
2. $S = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} \middle| \; x \text{ is a real number } \neq 0 \right\}$

   is a group under multiplication of matrices. Is it commutative?
3. Show that a group has exactly one idempotent element.
4. Let Z be the set of all integers. On Z define $*$ as follows: $a * b = a + b + 1$, where + is an ordinary addition. Show that $(Z, *)$ is a group.
5. In the set Z of all integers define $\oplus$ and $\odot$ by $a \oplus b = a + b - 1$ and $a \odot b = a + b - ab$. Prove that $(Z, \oplus, \odot)$ is a commutative ring.

3.5 LATTICES AND BOOLEAN ALGEBRA

1. Introduction

Recall a partial ordering relation and a **poset (partially ordered set)**. Throughout this section, unless stated otherwise, a partial ordering relation will be denoted by $\le$, a poset by $(P, \le)$; N stands for the set of all positive integers.

Quasi-Order: Let R be a relation on a non empty set A. Then R is said to be a **quasi-order** on A if,

(a) R is **irreflexive**; that is, $a \not R a$ for any $a \in A$

(b) R is transitive.

Comparability: Let a, b belong to $(P, \le)$. Then a and b are said to be **comparable**, if $a \le b$ or $b \le a$. a and b are **non comparable**, denoted by $a \parallel b$, if neither $a \le b$ nor $b \le a$.

Totally ordered set: If each pair of elements in P is comparable, then $(P, \le)$ is said to be **totally ordered** or **linearly ordered** and P is called a **chain**.

Example: Let N be partially ordered by divisibility. 3 and 15 are comparable since $3 | 15$. But 8 and 15 are not comparable since neither $8 | 15$ nor $15 | 8$. So N is not linearly ordered by divisibility. However, the subset $S = \{3, 6, 12, 48\}$ is linearly ordered since $3 | 6$, $6 | 12$, $12 | 48$.

Note that N with the usual order $\le$ (less than or equal to) is linearly ordered and so each subset of N is also linearly ordered.

Note: The symbols $\le, <, >, \ge$ are interpreted thus:

$a \le b \Leftrightarrow b \ge a$

$a \le b$ means **a precedes b**

$b \ge a$ means **b succeeds a**

$a < b$ means $a \le b$ and $a \ne b$; i.e., **a strictly precedes b**

$b > a$ means **b strictly succeeds a**

When $\le$ is a partial order on P then $\ge$ is also a partial order on P and is called the **dual order**.

Cover of an element: Let $a, b \in P$. Then a is said to be an **immediate predecessor of b** or b is said to be an **immediate successor** of a or **b is a cover of a**, denoted by $a << b$, if $a < b$ and no element of P lies between a and b. i.e., there is no element c in P satisfying $a < c$ and $c < b$.

Lexicographical ordering: Let A and B be linearly ordered sets. Then the **lexicographic ordering** or **dictionary order** on $A \times B$ is defined as follows:

$$(a, b) < (a', b') \text{ if } a < a' \text{ or if } a = a', b < b'$$

For example, let A = B = N = set of all positive integers. Then according to lexicographic ordering on $N \times N$,

$(5, 78) < (7, 1)$ since $5 < 7$

$(4, 6) > (4, 2)$ since $4 = 4$, $6 > 2$

$(5, 5) > (4, 23)$ since $5 > 4$

Hasse Diagrams of posets: Let $(P, \leq)$ be a finite poset. Hasse diagram of P is a directed graph. Its vertices are the elements of P. Elements x and y of P are connected by an edge (line) if y covers x (no arrow is used). Since the relation $\leq$ is anti-symmetric, Hasse diagram cannot contain any (directed) cycle.

Example: The Hasse diagram of P = {1, 2, 4, 5, 6, 25, 100} is shown in **Figure** below.

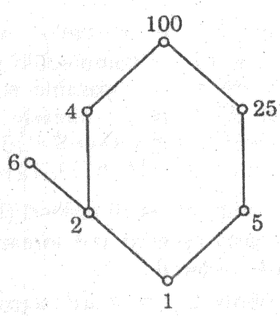


Fig. Poset diagram

Minimal element: An element x in P is a **minimal element**, if $y \leq x \Rightarrow y = x$. i.e., x is a minimal element if no other element in P strictly precedes (is less than) x.

Maximal element: An element x in P is a **maximal element**, if $y \geq x \Rightarrow y = x$. i.e., x is maximal element if no other element in P strictly succeeds (is greater than) x.

Least element: An element y in P is a **least element (or first element)** of P, if $y \leq x$ for all $x \in P$.

Greatest element: An element y in P is a **greatest element (or last element)** of P, if $y \geq x$ for all $x \in P$.

Note: A poset P may have neither a least element nor a greatest element. P can have atmost one least element which must be a minimal element. Similarly P can have atmost one greatest element which must be a maximal element.

For the poset in **Figure** above, least element is 1 which is also the minimal element; greatest element is 100 which is also the maximal element.

Example: Let P = {1, 2, 3, 4, 6, 8, 9, 12, 18, 24} and $a \leq b \Leftrightarrow a \mid b$. Then $(P, \leq)$ is a poset. For this poset (**Figure** below), 18 and 24 are maximal elements; but they are not below the greatest elements. 1 is the only minimal element, which is also the least element of P.

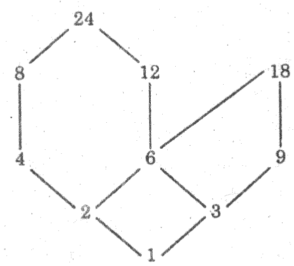


Fig. Chain

For the chain in **Figure**, least element is a; greatest element is e.

For the poset $(P(S), \leq)$, where S is a non-empty set, ϕ is the first (least) element whereas S is the last (greatest) element.

Lower Bound: Let A be a non-empty subset of $(P, \leq)$. An element x in P is said to be a **lower** bound of A, if $x \leq a$ for all $a \in A$.

Greatest Lower Bound (GLB): GLB of A ($A \subseteq P$) is x

(i) if x is a lower bound of A

(ii) if y is any other lower bound of A then $y \leq x$

Upper bound: Let A be a non-empty subset of $(P, \leq)$. An element y in P is said to be an **upper bound** of A, if $a \leq y$ for all $a \in A$.

Least Upper Bound (LUB): LUB of A ($A \subseteq P$) is y

(i) if y is an upper bound of A

(ii) if x is any other upper bound of A, then $y \leq x$

Sometimes GLB is called **infimum** and LUB is called **supremum**.

If A has an upper bound then we say that A is **bounded above** and A is **bounded below**, if A has a lower bound. If A has an upper and lower bound then A is said to be **bounded**.

Example: Let P = {u, v, w, x, y, z} have the

Hasse diagram (see **Figure**). Let A = {v, w, x}. Then the upper bounds of A are y, z since these are the only elements which succeed every element of A. LUB of A does not exist since y and z are non-comparable. The lower bounds of A are v, w and GLB of A is v.

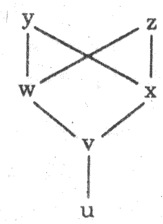


Fig.

Well-ordered Sets: A poset (P, ≤) is said to be well-ordered, if each subset of P has a **least element**.

For example, N with usual ≤ is a well-ordered

Note: 1. Every subset of a well-ordered set is well-ordered.

2. Any well-ordered set P is a chain. For, if a, b ∈ P then the subset {a, b} has a least element. Thus a and b are comparable.

3. Z (= set of all integers) with usual ≤ is a chain. But Z is not well-ordered since Z (which is an improper subset of Z) has no least element.

Problem: Let Q be the set of all rationals. Let A be the set of rational numbers lying between $\sqrt{2}$ and $\sqrt{3}$ on the real line. Find if LUB and GLB of A exist.

Solution: A = $\{x \mid 2 < x^2 < 3$, x is a rational$\}$. Then A has infinite number of upper and lower bounds. $\sqrt{2}$, $\sqrt{3}$ do not lie in Q and hence cannot be considered as lower and upper bounds of A. Thus LUB and GLB of A do not exist.

Problem: Let A = {2, 3, 4, 5 ... } be ordered by x ≤ y ⇔ x divides y. Find all the minimal and maximal elements of A.

Solution: If p is a prime then only p divides p

[∵ 1 ∉ A].

∴ all the prime numbers are minimal elements.

If a ∈ A is not a prime then there exists a number b ∈ A such that b divides a. So b ≤ a and b ≠ a.

∴ the only minimal elements are the primes.

There is no maximal element since for each a ∈ A there exists 2a ∈ A such that a divides 2a.

2. Lattices

Lattice: A lattice is a poset (P, ≤) in which any two elements x, y have a GLB and a LUB. The GLB of x, y is called the **meet** of x and y; it is denoted by $x \wedge y$. The LUB of x, y is called the **join** of x and y; it is denoted by $x \vee y$.

Example

1. N and Z with usual ≤ are lattices for which $x \wedge y$ = lesser of x, y; $x \vee y$ = greater of x, y.
2. The poset (P(S), ⊆) is a lattice with $A \wedge B = A \cap B$ and $A \vee B = A \cup B$. This lattice is known as power set lattice.
3. (N, ≤) where $a \leq b \Leftrightarrow a \mid b$ is a lattice. Here
   $a \wedge b$ = g.c.d (greatest common divisor) of a, b
   $a \vee b$ = l.c.m (least common multiple) of a, b.
   For example, $3 \wedge 5 = 1$, $3 \vee 5 = 15$
4. Any chain is a lattice in which
   $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$
5.

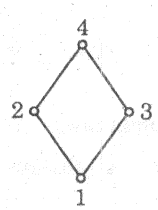


Fig. (a)

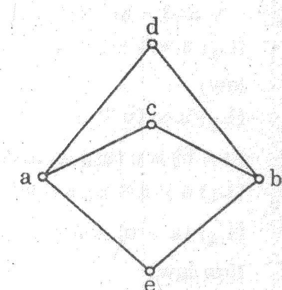


Fig. (b)

The poset in **Figure (a)** is a lattice; whereas the poset in **Figure (b)** is not a lattice, since a and b do not have a LUB. (c, d are upper bounds which are non comparable and so LUB of {a, b} does not exist).

Dual Lattice: For the lattice (P, ≤, ∨, ∧) the dual lattice is (P, ≥, ∧, ∨).

Figure (d) shows the dual lattice diagram of **Figure (c).**

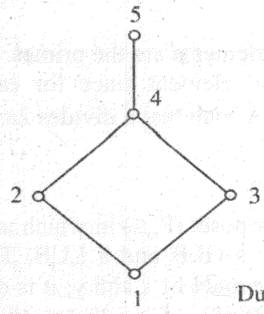


Fig. (c)

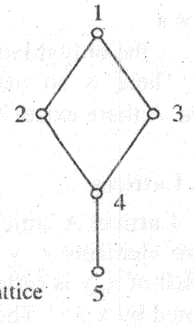


Fig. (d)

We can define lattices in two ways and show that one implies the other.

I. A lattice is a poset $(P, \leq)$ satisfying

(P_1) $x \leq x$

(P_2) $x \leq y, y \leq x \Rightarrow x = y$

(P_3) $x \leq y, y \leq z \Rightarrow x \leq z$

and any two of whose elements x and y have a greatest lower bound $(x \wedge y)$ and a least upper bound $(x \vee y)$.

II. L is any set in which two binary operations $\vee$ and $\wedge$ are defined to satisfy

(L_1) $a \vee b = b \vee a$; $a \wedge b = b \wedge a$ (commutative law)

(L_2) $a \vee (b \vee c) = (a \vee b) \vee c$; $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (associative law)

(L_3) $a \vee a = a$; $a \wedge a = a$ (idempotent law)

(L_4) $(a \vee b) \wedge a = a$; $(a \wedge b) \vee a = a$ (absorption law)

Then L is a lattice relative to a suitable definition of order $\leq$ and $\vee$, $\wedge$ are the join and meet of this lattice.

Problem: If L is a set, satisfying L_1 and L_4 show that for a, b $\in$ L, $a \vee b = a$ and $a \wedge b = b$ are equivalent.

Solution: **(i)** Suppose $a \vee b = a$, Then

$$a \wedge b = (a \vee b) \wedge b = (b \vee a) \wedge b \text{ (by } L_1)$$
$$= b \text{ (by } L_4)$$

$\therefore a \vee b = a \Rightarrow a \wedge b = b.$

(ii) Suppose $a \wedge b = b$. Then

$$a \vee b = a \vee (a \wedge b) = (a \wedge b) \vee a \text{ (by } L_1)$$
$$= a \text{ (by } L_4)$$

Thus $a \wedge b = a \Rightarrow a \vee b = a$

Problem: Prove that $L_4 \Rightarrow L_3$

Solution: Let a, b $\in$ L. Then

$$a \vee a = a \vee [a \wedge (a \vee b)] \text{ by } L_4$$
$$= a \vee (a \wedge x) \text{ where } x = a \vee b \in L$$
$$= a \text{ by } L_4$$

Thus $a \vee a = a$. Writing the dual of this statement (by interchanging $\vee$ and $\wedge$) we get, $a \wedge a = a$. Thus L_3 follows from L_4.

Problem: Let L be a set for which L_1, L_2, L_3, L_4 are true. Define a relation $\leq$ in L by $a \leq b \Leftrightarrow a \wedge b = a$ or $a \vee b = b$. Show that $(L, \leq, \vee, \wedge)$ is a lattice.

Solution: First we show that $(L, \leq)$ is a poset.

By L_3, $a \wedge a = a$ $\therefore$ by definition of $\leq$, $a \leq a$

i.e., $\leq$ is reflexive.

Suppose, $a \leq b$ and $b \leq a$.

Then, $a \wedge b = a$ and $b \wedge a = b$

By L_1, $a \wedge b = b \wedge a$ $\therefore$ $a = b$ i.e., $\leq$ is anti symmetric.

Suppose, $a \leq b$ and $b \leq c$. Then

$a \wedge b = a$ and $b \wedge c = b$

Now, $a \wedge c = (a \wedge b) \wedge c$

$= a \wedge (b \wedge c)$ by L_2

$= a \wedge b$

$= a$ so that $a \leq c$

Thus $\leq$ is transitive also.

$\therefore$ $(L, \leq)$ is a poset.

Let a, b $\in$ L.

By L_4, $(a \wedge b) \vee a = a$ $\therefore$ $a \wedge b \leq a$

Now $a \wedge b = a \wedge (b \wedge b)$ by L_3

$= (a \wedge b) \wedge b$ by L_2

$\therefore$ $(a \wedge b) \wedge b = a \wedge b$ and hence $a \wedge b \leq b$.

Thus $a \wedge b$ is such that it is $\leq a$ and $\leq b$.

$\therefore$ it is a lower bound of a, b.

Let c be a lower bound of a, b. Then $c \leq a$ and $c \leq b$. i.e., $c \wedge a = c$, $c \wedge b = c$

Now $c \wedge (a \wedge b) = (c \wedge a) \wedge b$ by L_2

$$= c \wedge b$$
$$= c$$

$\therefore c \leq a \wedge b$

Thus $a \wedge b$ is the greatest lower bound of a, b. Similarly, we can show that $a \vee b$ is the least upper bound of a, b.

$\therefore (L, \leq, \vee, \wedge)$ is a lattice.

Problem: $(P, \leq)$ is a poset in which any two elements x and y have a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$. Show that $\wedge$ and $\vee$ satisfy L_1, L_2, L_3 and L_4.

Solution: (i) Greatest lower bound (g.l.b) and least upper bound (l.u.b) are symmetric functions of their arguments.

$\therefore a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ and so L_1 is true.

(ii) $(a \vee b) \vee c$ is the l.u.b of a, b, c and l.u.b is unique.

$\therefore (a \vee b) \vee c = a \vee (b \vee c)$

Similarly, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

i.e., L_2 is true.

(iii) Clearly $a \vee a = a$, $a \wedge a = a$. $\therefore L_3$ holds

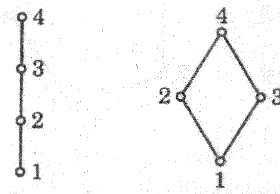
(iv) since $a \leq a \vee b$, $(a \vee b) \wedge a = a$

Similarly $(a \wedge b) \vee a = a$ $\therefore L_4$ is true.

Problem: Show that a lattice L with less than four elements is a chain. Is this statement true in the case of lattices with 4 elements or 5 elements?

Solution: Obviously L is a chain when it has 1 or 2 elements. **Figure A** shows the below diagrams of posets of which (a), (b) and (d) alone are lattices (in fact, chains); others are not lattices as we can find a pair of elements without a g.l.b. or a l.u.b.

There are two lattices with 4 elements, only one is a chain. (**Figure** below)



(h) (i)

Fig.

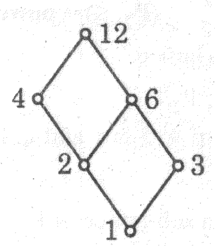
For a 5 element lattice, the lattice diagrams are as shown in Figure B; only one of these is a chain.

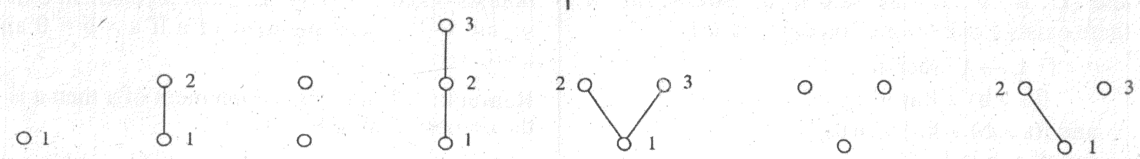
Sub-lattice: Let $(L, \leq, \vee, \wedge)$ be a lattice. A subset S of L is said to be sub-lattice of L if S is closed with respect to $\vee$ and $\wedge$. That is, for all a, b in S, $a \vee b$, $a \wedge b$ belong to S. Note that every sub-lattice is itself a lattice.

Example: Let D_{12} = set of all positive divisors of 12.

$$= \{1, 2, 3, 4, 6, 12\}$$

Then D_{12} is a sub-lattice of the lattice $(N, \leq)$ where $a \leq b \Leftrightarrow a \mid b$





(a) (b) (c) (d) (e) (f) (g)

Fig. A

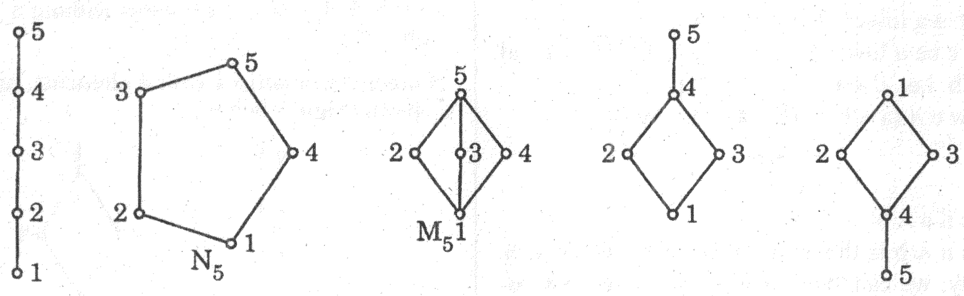


Fig. B

Problem: Show that the poset in **Figure (a)** is a lattice but it is not a sub-lattice of the lattice of the power set of three elements in **Figure (b)**.

Solution: In the poset $(P_1, \leq)$ any two elements have a unique g.l.b and a l.u.b.

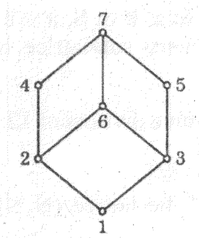


Fig. (a)
$(P_1, \leq)$

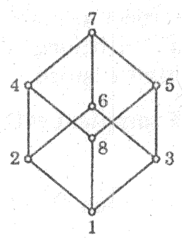


Fig. (b)
$(P_2, \leq)$ = power set lattice

$\therefore (P_1, \leq)$ is a lattice.

Clearly $P_1 \subseteq P_2$.

In P_1, g.l.b of 4, 5 is 1 and in P_2, g.l.b of 4, 5 is $8 \notin P_1$.

$\therefore P_1$ is not a sub-lattice of P_2.

Isomorphic Lattices: Two lattices $(L, \leq, \vee, \wedge)$ and $(L', \leq', \vee', \wedge')$ are said to be **isomorphic** if there exists a one-to-one correspondence,

$f : L \to L'$ such that

$f(a \vee b) = f(a) \vee' f(b)$

and $f(a \wedge b) = f(a) \wedge' f(b)$

for all a, b in L.

Complete Lattice: A lattice L is said to be a **complete lattice** if every non-empty subset of L has a least upper bound and a greatest lower bound in L.

Examples of complete lattice are

(i) any finite lattice and

(ii) the power set lattice $(P(S), \subseteq)$.

Bounds: A lattice $(L, \leq, \vee, \wedge)$ is said to have a **lower bound** 0, if $0 \leq x$ for all x in L; L is said to have an **upper bound** 1, if $x \leq 1$ for all x in L.

L is said to be a **bounded lattice**, if L has both 0 and 1. In a bounded lattice.

$a \vee 0 = a, a \wedge 0 = 0,$

$a \vee 1 = 1, a \wedge 1 = a$ for any a in L.

Example: (i) $N' = N \cup \{0\}$ under $<$ has a lower bound 0 but no upper bound.($\because 0 < 1 < 2 < 3 ....$)

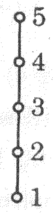
(ii) Let E = universal set. Then the lattice $(P(E), \leq)$ is a bounded lattice with E as 1 and ϕ as 0.

(iii) For the finite lattice $L = \{a_1, a_2, ... a_n\}$, $a_1 \vee a_2 \vee .... \vee a_n$ and $a_1 \wedge a_2 \wedge .... \wedge a_n$ are the upper and lower bounds respectively. Thus **every finite lattice is bounded.**

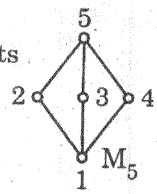
Complement: Let $(L, \vee, \wedge)$ be a lattice with bounds 0 and 1. Let $a \in L$. Then an element b in L is said to be a **complement of a** if $a \wedge b = 0$ and $a \vee b = 1$.

Remark: 1. If b is the complement of a then a is the complement of b.

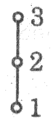
2. In the lattice N_5 of **Figure** below, 4 has two complements 2 and 3. Thus complement of an element need not be unique.



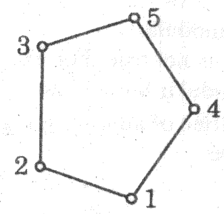
The elements 2, 3, 4 have no complements



M_5



Chain



N_5

Fig.

3. In the chain, refer **Figure** above.

∴ complements need not exist

Complemented Lattice: A lattice L is said to be **complemented** if **(i)** L is bounded and **(ii)** each element in L has a complement.

The lattice M_5 of **Figure** above is a complemented lattice; but the chain of **Figure** above is not a complemented lattice since $2 \wedge 3 = 2$ which is not the lower bound 1.

Distributive Lattice

A lattice $(L, \vee, \wedge)$ is said to be **distributive** if

$$\left\{\begin{array}{l}(\mathbf{i})\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ (\mathbf{ii})\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\end{array}\right\}$$

(Distributive Law)

for all a, b, c in L. Otherwise L is said to be non-distributive.Consider the lattice of **Figure (a).** It is non-distributive. For,

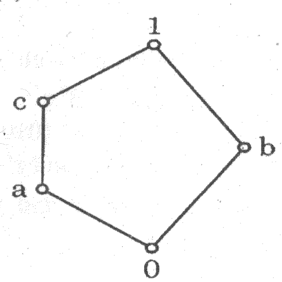


Fig. (a)

$a \vee (b \wedge c) = a \vee 0 = a$ and

$(a \vee b) \wedge (a \vee c) = 1 \wedge c = c$

$\therefore a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

It is easy to check that the lattice M_5 of **Figure (b)** is also non-distributive.

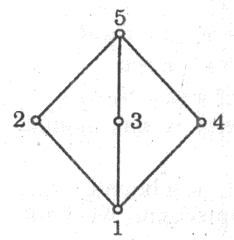


Fig. (b)

Remark: 1. Every chain is a distributive lattice

2. In a distributive lattice

$a \wedge b = a \wedge c,\ a \vee b = a \vee c \Rightarrow b = c$ (cancellation law)

For, $b = b \vee (a \wedge b)$ (Absorption law)

$= b \vee (a \wedge c)$

$= (b \vee a) \wedge (b \vee c)$ ($\because$ L is distributive)

$= (c \vee a) \wedge (b \vee c)$

$= (c \vee a) \wedge (c \vee b)$

$= c \vee (a \wedge b)$ ($\because$ L is distributive)

$= c \vee (a \wedge c)$

$= c \vee (c \wedge a)$

$= c$ (Absorption law)

Modular Lattice: A lattice L is said to be **modular**, if $a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

Remark: 1. Every distributive lattice is modular. The converse is not true.

For, let L be distributive. Let a, b, c ∈ L with ≤ c. Then

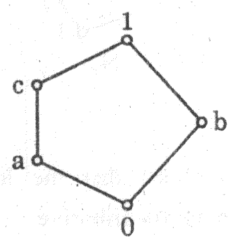
$a \vee (b \wedge c) = (a \vee b) \wedge (a \wedge c)$ ($\because$ L is distributive)

$= (a \vee b) \wedge c \; (\because a \le c \Rightarrow a \wedge c = c)$

$\therefore$ L is modular.

Converse is not true. For example, the lattice of **Figure** is modular but it is non-distributive.

2. The lattice of adjoining **Figure** is non-modular.



For, here $a \le c$.

$a \vee (b \wedge c) = a \vee 0 = a$ and

$(a \vee b) \wedge c = 1 \wedge c = c$

$\therefore a \vee (b \wedge c) \ne (a \vee b) \wedge c$

i.e., the lattice is non-modular; it is non-distributive too.

Problem: Let L be a bounded distributive lattice. Show that complements, when they exist, are unique.

Solution: Suppose $a \in L$ has two complements b and c. Then

$$a \vee b = 1, \quad a \wedge b = 0$$
$$a \vee c = 1, \quad a \wedge c = 0$$

Now $b = b \vee 0 = b \vee (a \wedge c)$
$= (b \vee a) \wedge (b \vee c)$ ($\because$ L is distributive)
$= 1 \wedge (b \vee c)$
$= b \vee c$

Similarly, $c = c \vee 0 = c \vee (a \wedge b)$
$= (c \vee a) \wedge (c \vee b)$
$= 1 \wedge (c \vee b)$
$= c \vee b$

Thus $b = b \vee c = c \vee b = c$.

3.6 BOOLEAN ALGEBRA

A non-empty set B together with two binary operations denoted by $\oplus$ and $*$, a unary # operation $'$ and two distinct elements 0 and 1 is called a **Boolean algebra** if the following axioms hold for any elements a, b, c in B:

(B_1) $a \oplus b = b \oplus a$, $a * b = b * a$ (commutative law)

(B_2) $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

$a * (b \oplus c) = (a * b) \oplus (a * c)$ (Distributive law)

(B_3) $a \oplus 0 = a$, $a * 1 = a$ (Identity law)

(B_4) $a \oplus a' = 1$, $a * a' = 0$ (Complement law)

0 is called the **zero** element, 1 is the **unit** element and a' is the **complement** of a. The operations $\oplus$, $*$, $'$ are called **sum, product** and **complement** respectively. Incorporating all these, we find that a Boolean algebra is specified by $(B, \oplus, *, ', 0, 1)$.

Example: Let B = {0, 1} = set of **bits** (binary digits). Define $\oplus$, $*$, $'$ on B by

$\oplus$	0	1
0	0	1
1	1	1

$*$	0	1
0	0	0
1	0	1

a	a′
0	1
1	0

Table

Then B is a Boolean algebra.

Example : Let $B^n = \underbrace{B \times B \times \ldots\ldots \times B}_{\text{n times}}$

Define $\oplus$, $*$, $'$ componentwise using **table** above.

For instance, consider B^5 with x = 11010, y = 10101 (5-bit sequences in B^5).

Then $x \oplus y = 11010 \oplus 10101 = 11111$, $x * y = 11010 * 10101 = 10000$ and $x' = 00101$.

The zero element 0 = 00000 and unit element 1 = 11111.

Example: Let F = Collection of sets closed under the operations of union, intersection and complementation. Then F is a Boolean algebra for which zero element is ϕ, unit element is the universal set E.

Example: Let D_{12} = set of all positive divisors of 12.
= {1, 2, 3, 4, 6, 12}

A **unary** operation on a set A is a function from A into A. For example, the absolute value function $f : Z \to Z$, $f(n) = |n|$ is a unary operation on Z (set of all integers).

Define $\oplus, *, '$ on D_{12} by

$a \oplus b$ = l.c.m of a, b

$a * b$ = g.c.d of a, b

$a' = \frac{12}{a}$

Then D_{12} is a Boolean algebra with 1 as zero element and 12 as the unit element.

Isomorphic Boolean algebras

Two Boolean algebras B and B′ are said to be **isomorphic**, if there is a one-to-one correspondence $f : B \to B'$ which preserves all the three operations. i.e.,

$f(a \oplus b) = f(a) \boxplus f(b)$

$f(a * b) = f(a) * f(b)$

$f(a)' = [f(a)]'$ a, b $\in$ B

where $\oplus, *, '$ are operations for B; $\boxplus$ $\oplus$, $*$, $'$ are operations for B′.

Duality

The **dual** of any statement in a Boolean algebra is got by interchanging $\oplus$ and * and interchanging 0 and 1 in the given statement.

The following results can be deduced from (B_1) to (B_4). Let a, b, c $\in$ B.

1. $a \oplus a = a, \quad a * a = a$ (Idempotent law)

2. $a \oplus 1 = 1, \quad a * 0 = 0$ (Boundedness law)

3. $a \oplus (a * b) = a, a * (a \oplus b) = a$ (Absorption law)

4. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,
$a * (b * c) = (a * b) * c$ (Associative law)

5. $a \oplus x = 1, \ a * x = 0 \Rightarrow x = a'$ (Uniqueness of complement)

6. $0' = 1, 1' = 0$

7. $(a')' = a$ (Involution law)

8. $(a \oplus b)' = a' * b'$
$(a * b)' = a' \oplus b'$ (De Morgan's law)

Thus we find that a Boolean algebra B satisfies the commutative, associative and absorption laws. So it is a lattice for which $\oplus$ and $*$ are the join and meet operations respectively. For this lattice,

$a \oplus 1 = 1 \Rightarrow a \le 1$ and

$a * 0 = 0 \Rightarrow 0 \le a$ for $a \in B$

Thus B is also bounded.

Also, B is distributive and complemented (by (B_2) and (B_4)). Thus a **Boolean algebra is a (bounded) complemented distributive lattice.**

Problem: In a Boolean algebra the following are equivalent.

(i) $a \oplus b = b$

(ii) $a * b = a$

(iii) $a' \oplus b = 1$

(iv) $a * b' = 0$

Solution: (i) $\Rightarrow$ (ii)

Suppose $a \oplus b = b$. Then

$a * b = a * (a \oplus b) = a$ (Absorption law)

(ii) $\Rightarrow$ (i)

Suppose $a * b = a$. Then

$a \oplus b = b \oplus a$ (by (B_1))

$= b \oplus (a * b)$

$= b \oplus (b * a)$ (by (B_1))

$= b$ (Absorption law)

(i) $\Rightarrow$ (iii)

Suppose $a \oplus b = b$. Then

$a' \oplus b = a' \oplus (a \oplus b) = (a' \oplus a) \oplus b$

$= (a \oplus a') \oplus b = 1 \oplus b = 1$

(iii) $\Rightarrow$ (i)

Suppose $a' \oplus b = 1$. Then

$a \oplus b = 1 * (a \oplus b) = (a' \oplus b) * (a \oplus b)$

$= (a' * a) \oplus b$ (by Distributive law)

$= 0 \oplus b$

$= b$

(iii) $\Rightarrow$ (iv)

Suppose $a' \oplus b = 1$. Then

$0 = 1' = (a' \oplus b)' = a'' * b'$ (by De Morgan's law)

$= a * b'$

(iv) $\Rightarrow$ (iii)

Suppose $a * b' = 0$. Then

$1 = 0' = (a * b')' = a' \oplus b'' = a' \oplus b$

Thus all the four are equivalent.

Problem: Prove the following Boolean identities:

(a) $a \oplus (a' * b) = a \oplus b$

(b) $a * (a' \oplus b) = a * b$

(c) $(a * b) \oplus (a * b') = a$

Solution

(a) $a \oplus (a' * b) = (a \oplus a') * (a \oplus b)$

$= 1 * (a \oplus b)$

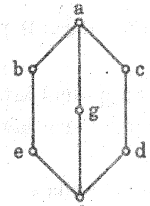
$= a \oplus b$

(b) It is the dual statement of (a).

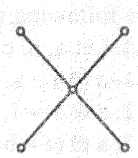
(c) $(a * b) \oplus (a * b') = a * (b \oplus b') = a * 1 = a$

OBJECTIVE QUESTIONS

1. Let $N \times N$ be ordered lexicographically. Then
   (A) $(1, 3) < (1, 2)$ (B) $(1, 3) = (1, 2)$
   (C) $(1, 3) > (1, 2)$ (D) none of these
2. Let $A = \{2, 3, 4, 5, 6, 8, 9, 10\}$ and $x \le y \Leftrightarrow x$ is a multiple of y. Then A has
   (A) one minimal element
   (B) two minimal elements
   (C) three minimal elements
   (D) four minimal elements
3. If a and b are minimal elements in a totally ordered set, then
   (A) $a \le b$
   (B) $a \ge b$
   (C) $a = b$
   (D) a and b are not comparable
4. Hasse diagrams
   (A) are drawn for Boolean algebras
   (B) can contain any (directed) cycle
   (C) are undirected graphs
   (D) are drawn for finite partially ordered sets
5. Every finite lattice
   (A) has a least element but no greatest element
   (B) has a greatest element but no least element
   (C) has neither a least element nor a greatest element
   (D) has a least element and a greatest element
6. For any lattice,
   (A) $a \vee (b \wedge c) \ge (a \vee b) \wedge (a \vee c)$
   (B) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
   (C) $a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$
   (D) none of these
7. The dual of the Boolean expression $(a \oplus 1)(a \oplus 0) = a$ is
   (A) $(a * 1)(a * 0) = a$
   (B) $(a * 1) \oplus (a * 0) = a$
   (C) $(a * 0) \oplus (a * 1) = a$
   (D) $(a * 0)(a * 1) = a$
8. Which of the following statements is true?
   (A) The closed interval $[a, b] = \{x \in L | a \le x, x \le b\}$ is a chain.
   (B) Any modular lattice is distributive.
   (C) There is a lattice with four elements which is not a chain.
   (D) In a lattice $(L, \ge, \oplus, *)$
   $a \ge b$ and $b \ge c \not\Rightarrow a \ge b \oplus c$.
9. If a linearly ordered set S has only one maximal element a, then
   (A) a is a first element
   (B) a is a last element
   (C) a is neither a first element nor a last element
   (D) nothing can be said about a
10. Which of the following statements is false?
    (A) In a Boolean algebra, $a \le b \Leftrightarrow a * b' = 0$ where b' is a complement of b.
    (B) Let $N = \{1, 2, 3, ...\}$ be ordered by divisibility. Then the set $\{3, 15, 5\}$ is linearly ordered.
    (C) The least element 0 in a bounded lattice has the unique complement 1.
    (D) Every chain is a distributive lattice.
11. Which of the following is **not** a lattice?
    (A)



    (B)



    (C) Power set of any non-empty set partially ordered by set inclusion.
    (D) The collection R(S) of all subsets of a set S such that
    $A, B \in R(S) \Rightarrow A \cap B, A \cup B \in R(S)$

KEY

1. (C) **2.** (D) 6, 8, 9, 10 are minimal elements.

3. (C) **4.** (D) **5.** (D) **6.** (C) **7.** (C)

8. (C) **9.** (B) **10.** (B) **11.** (B)

3.7 PRINCIPLE OF MATHEMATICAL INDUCTION

Introduction

Let $\{P(n)\}$ be a set of statements one for each natural number n. The **principle of mathematical induction** says that if we prove

(1) $P(1)$ is true

(2) $P(k)$ is true $\Rightarrow P(k+1)$ is true (k, any natural number) then $P(n)$ is true for all natural numbers n.

Proof by the method of induction can be compared with climbing an infinite ladder. To climb such a ladder, we have to climb the first rung of the ladder and having climbed anyone rung the next rung also should be climbed.

(1) is called the **basis of induction**; **(2)** is called the **induction step**. The assumption $P(k)$ is true $\Rightarrow$ $P(k+1)$ is true is known as **induction hypothesis**.

P(n) can be statements of the form,

(i) if n is even, then n + 2 is also even

(ii) $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

(iii) $3.5^{2n+1} + 2^{3n+1}$ is divisible by 17.

Problems with Solutions

Problem: Prove by induction

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Solution: Let $P(n)$: $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

When n = 1, L.H.S (left hand side) = 1 and R.H.S (right hand side) = $\dfrac{1(1+1)}{2} = 1$.

$\therefore$ P(1) is true.

Assume that P(k) is true

i.e., $1 + 2 + \cdots + k = \dfrac{k(k+1)}{2}$ (induction hypothesis) ... (1)

Now, $1 + 2 + \cdots + (k+1) = [1 + 2 + \cdots + k] + [k+1]$

$$= \frac{k(k+1)}{2} + (k+1) \quad \text{by (1)}$$

$$= \frac{(k+1)}{2}(k+2)$$

$$= \frac{(k+1)(k+1+1)}{2}$$

$\therefore$ P(k + 1) is true.

Thus, by induction principle, P(n) is true for all natural numbers n.

i.e., $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

Problem: Prove by induction

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{when } x < 1$$

Solution: Let $P(n)$: $1 + x + x^2 + \cdots + x^n$

$$= \frac{1 - x^{n+1}}{1 - x}, \; x < 1$$

When n = 1, L.H.S = 1 + x, R.H.S

$$= \frac{1 - x^2}{1 - x} = 1 + x$$

$\therefore$ P(1) is true.

Assume that P(k) is true.

i.e., $1 + x + x^2 + \cdots + x^k = \dfrac{1 - x^{k+1}}{1 - x}$ ... (2)

Now, $1 + x + x^2 + \cdots + x^{k+1} = (1 + x + x^2 + \ldots + x^k) + x^{k+1}$

$$= \frac{1 - x^{k+1}}{1 - x} + x^{k+1} \quad \text{[by (2)]}$$

$$= \frac{1 - x^{k+1} + x^{k+1}(1 - x)}{1 - x}$$

$$= \frac{1 - x^{k+2}}{1 - x}$$

$$= \frac{1 - x^{(k+1)+1}}{1 - x}$$

This shows that P(k + 1) is true. $\therefore$ by induction principle, P(n) is true for all natural numbers n.

Problem: Prove by induction

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

Solution: Let $P(n) : 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$

When n = 1, L.H.S. $= 1 \cdot 1! = 1$ and R.H.S. $= (1+1)! - 1 = 1$

So P(1) is true.

Assume that P(k) is true.

i.e., $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$ ... (3)

Now $1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1) \cdot (k+1)!$

$= [1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k!] + (k+1) \cdot (k+1)!$

$= (k+1)! - 1 + (k+1)(k+1)!$ [by (3)]

$= (k+1)! [1 + k + 1] - 1$

$= (k+1)!(k+2) - 1$

$= (k+2)! - 1$

$= (k+1+1)! - 1$

This shows that P(k + 1) is true.

$\therefore$ by induction principle, P(n) is true for all natural numbers n.

Problem: Prove that $5^n - 1$ is divisible by 4 for $n \geq 1$.

Solution: Let P(n): $5^n - 1$ is divisible by 4, $n \geq 1$.

When n = 1, $5^n - 1 = 5 - 1 = 4$ which is divisible by 4.

$\therefore$ P(1) is true.

Assume that P(k) is true, i.e., $5^k - 1$ is divisible by 4 (induction hypothesis).

When $n = k+1$, $5^n - 1 = 5^{k+1} - 1$

$= 5 \cdot 5^k - 1$

$= (4+1) 5^k - 1$

$= 4 \cdot 5^k + (5^k - 1)$

$4 \cdot 5^k$ is divisible by 4; $5^k - 1$ is divisible by 4 (by induction hypothesis)

$\therefore$ P(k + 1) is true.

Thus, by induction principle, $5^n - 1$ is divisible by 4 for $n \geq 1$.

Problem: Let P(n) be the statement

$$1 + 3 + 5 + \cdots + (2n-1) = n^2 + 5.$$

Show that P(k) is true $\Rightarrow$ P(k + 1) is true. Can we conclude that P(n) is true for all n?

Solution: Assume that P(k) is true.

i.e., $1 + 3 + 5 + \cdots + (2k-1) = k^2 + 5$ ... (1)

Then, $1 + 3 + 5 + \cdots + [2(k+1) - 1]$

$= [1 + 3 + \cdots + (2k-1)] + (2k+1)$

$= (k^2 + 5) + 2k + 1$ [by (1)]

$= (k+1)^2 + 5$

$\therefore$ P(k + 1) is true.

When n = 1, L.H.S = 1 and R.H.S $= 1^2 + 5 = 6$

i.e., P(1) is not true. Hence we cannot conclude that P(n) is true for all n.

Problem: Let P(n): 2 divides $2n - 1$,

(a) Prove that P(k) implies P(k + 1).

(b) Show that P(n) is not true for any n.

(c) Does the result (b) contradict induction hypothesis?

Solution: **(a)** Assume that P(k) is true. That is, 2 divides $(2k - 1)$.

Since 2 divides $2k - 1$, 2 divides $(2k - 1 + 2)$ also. i.e., 2 divides $(2k + 1)$.

When $n = k + 1$, $2n - 1 = 2(k+1) - 1 = 2k + 1$.

Thus, we find that P(k + 1) is true.

(b) Obvious since $2n - 1$ is always odd.

(c) The result (b) contradicts the induction hypothesis (namely, P(k) is true).

OBJECTIVE QUESTIONS

1. Decide if the following statements are True or False.

(a) It is possible to prove by induction that the power set of an 'n' element set has 2^n elements.

(b) $7^n - 1$ is divisible by 6 for n = 1, 2, ...

(c) $11^n - 6$ is divisible by 5, for $n \geq 1$

(d) $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by 17, $n \geq 1$

2. Choose the correct alternative:

(A) $2^n > n^3$ for

(a) $n \geq 8$ **(b)** $n > 10$

(c) $n \geq 10$ **(d)** $n \geq 11$

(B) $2^n + 2^{n+1} + \cdots + 2^{2n}$ equals

(a) $2^{n+1} - 2^n$ **(b)** $2^{2n+1} - 1$

(c) $2^{2n+2} - 2$ **(d)** $2^{2n+1} - 2^n$

(C) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1)$ has the value

(a) $\dfrac{n(n+1)(2n+1)}{6}$

(b) $\dfrac{n(n+1)(n+2)}{3}$

(c) $\dfrac{n(n+1)(2n+1)}{3}$

(d) $\dfrac{n(n+1)(n+2)}{6}$

KEY

1. **All the statements are True.**
2. **(A)** (c) **(B)** (d) **(C)** (b)

EXERCISE

1. Use induction to prove the following.

 (i) $1^2 + 2^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

 (ii) $1^2 + 3^2 + ... + (2n+1)^2 = \dfrac{(n+1)(2n+1)(2n+3)}{6}$ for $n \geq 1$

 (iii) 5 divides $n^5 - n$ whenever n is a non-negative integer.

 (iv) $\sum_{r=1}^{n} r^3 = \dfrac{n^2(n+1)^2}{4}$

 (v) $2^n > n^3$ for $n > 9$

2. Is $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots\cdots + \dfrac{1}{(n-1)n} = \dfrac{3}{2} - \dfrac{1}{n}$ correct? Justify.

3. Write a formula for the sum of the first n even positive integers. Prove it by mathematical induction.

PROBLEMS WITH SOLUTIONS

1. If + denotes symmetric difference Δ, prove

 (i) $A + (B + C) = (A + B) + C$ for any three sets A, B, C

 (ii) $A + A = \phi$ and $A + \phi = A$ for any set A

Solution

(i) First note that

(a) $A \Delta B = (A \cup B) - (A \cap B)$

(b) $A - (B - C) = (A - B) \cup (A \cap C)$

(c) $(A - B) - C = A - (B \cup C)$

(d) $(A \cup B) - (A \cup C) = B - (A \cup C)$

Now, $A + (B + C) = A \Delta (B \Delta C)$

$= [A - (B \Delta C)] \cup [(B \Delta C) - A]$ (by definition) ... (1)

Also, $A - (B \Delta C)$

$= A - [(B \cup C) - (B \cap C)]$ (by (a))

$= [A - (B \cup C)] \cup (A \cap B \cap C)$ (by (b)) ... (2)

Again, $(B \Delta C) - A$

$= [(B \cup C) - (B \cap C)] - A$ (by (a))

$= (B \cup C) - [(B \cap C) \cup A]$ (by (c))

$= (B \cup C) - [(A \cup B) \cap (A \cup C)]$ (using distributive law)

$= (B \cup C) - [(B \cup A) \cap (C \cup A)]$

$= [(B \cup C) - (B \cup A)] \cup [(C \cup B) - (C \cup A)]$ (by De Morgan's law)

$= [C - (B \cup A)] \cup [B - (C \cup A)]$ (by (d)) ...(3)

Using (2) and (3) in (1), we have

$A \Delta (B \Delta C) = (A \cap B \cap C) \cup [A - (B \cup C)] \cup [B - (C \cup A)] \cup [C - (A \cup B)]$ ... (4)

Interchanging A and C in (4),

$C \Delta (B \Delta A) = (C \cap B \cap A) \cup [C - (B \cup A)] \cup [B - (A \cup C)] \cup [A - (C \cup B)]$ ... (5)

But $C \Delta (B \Delta A) = (B \Delta A) \Delta C = (A \Delta B) \Delta C$

Since the right sides of (4) and (5) are equal, it follows that $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

i.e., $A + (B + C) = (A + B) + C$

(ii) $A + A = A \Delta A = A \cup A - A \cap A = A - A = \phi$

and $A + \phi = A \Delta \phi$

$= A \cup \phi - A \cap \phi = A - \phi = A.$

2. Show that for any three sets A, B, C

(i) $(A - B) - C = (A - C) - (B - C)$

(ii) $A \cap (B - C) = (A \cap B) - (A \cap C)$

Solution

(i) $(A - B) - C = \{x | x \in A - B \text{ and } x \notin C\}$

$= \{x | x \in A \text{ and } x \notin B \text{ and } x \notin C\}$

$= \{x | (x \in A \text{ and } x \notin C) \text{ and } x \notin B\}$

$= \{x | x \in (A - C) \text{ and } x \notin (B - C)\}$

$(\because x \notin B \Rightarrow x \notin B - C)$

$= \{x | x \in (A - C) - (B - C)\}$

$= (A - C) - (B - C).$

(ii) $A \cap (B - C) = \{x | x \in A \text{ and } x \in B - C\}$

$= \{x | x \in A \text{ and } (x \in B \text{ and } x \notin C)\}$

$= \{x | (x \in A \text{ and } x \in B) \text{ and } x \notin C\}$

$= \{x | x \in (A \cap B) \text{ and } x \notin (A \cap C)\}$

$(\because x \notin C \Rightarrow x \notin A \cap C)$

$= \{x | x \in (A \cap B) - (A \cap C)\}$

$= (A \cap B) - (A \cap C).$

3. Draw Venn diagrams for

(a) $A \cup B \subseteq B$, but $B \not\subseteq A$

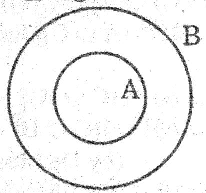
(b) $A \subseteq B, A \subseteq C, B \cap C \subseteq A$.

Solution

(a) $B \subseteq A \cup B$ always. Given $A \cup B \subseteq B$

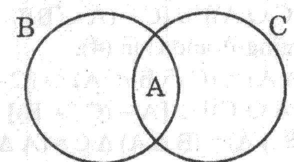
$\therefore B \subseteq A \cup B \subseteq B$. Thus, $A \cup B = B$

$\therefore A \subseteq B$ given $B \not\subseteq A$



(b) $A \subseteq B, A \subseteq C \Rightarrow A \subseteq B \cap C$

given $B \cap C \subseteq A$. $\therefore A = B \cap C$



4. Let A, B, C be subsets of the universal set U.

(i) If $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C$, show that $B = C$.

(ii) If $A \cap C \subseteq B \cap C$ and $A \cap \bar{C} \subseteq B \cap \bar{C}$, show that $A \subseteq B$.

Solution

(i) $B = U \cap B = (A \cup \bar{A}) \cap B = (A \cap B) \cup (\bar{A} \cap B)$

$= (A \cap C) \cup (\bar{A} \cap C)$

$(\because A \cap B = A \cap C, \bar{A} \cap B = \bar{A} \cap C)$

$= (A \cup \bar{A}) \cap C = U \cap C = C$

(ii) $A \cap C \subseteq B \cap C$ and $A \cap \bar{C} \subseteq (B \cap \bar{C})$

$\Rightarrow (A \cap C) \cup (A \cap \bar{C}) \subseteq (B \cap C) \cup (B \cap \bar{C})$

$\Rightarrow A \cap (C \cup \bar{C}) \subseteq B \cap (C \cup \bar{C})$

$\Rightarrow A \cap U \subseteq B \cap U$

$\Rightarrow A \subseteq B$.

5. Let $A = \{1, 2\}$ and $C = \{(1, 2), (1, 4), (2, 4)\}$

(i) Find a set B, so that $A \times B \supseteq C$.

(ii) Does there exists a set B, such that $A \times B = C$?

Solution

(i) $B = \{2, 4\}$

(ii) Suppose, there is a set B having n elements. Since A has 2 elements $A \times B$ will contain 2n elements. But C has 3 elements. Since $2n = 3$ is impossible for any positive integer n, there is no set B satisfying $A \times B = C$.

6. Let $A = B = \{1, 2, 3, 4, 8\}$. Define ~ by $a \sim b \Leftrightarrow a + 1 = b$. Find the range and domain of ~. Is ~ an equivalence relation, a partial ordering on the domain of ~?

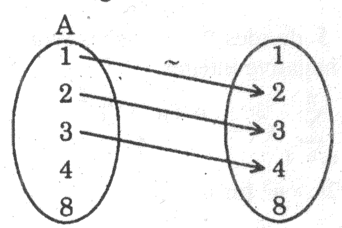
Solution

$a \sim b \Leftrightarrow a + 1 = b$

$\therefore \sim = \{(1, 2), (2, 3), (3, 4)\}$ so that domain of~ $= \{1, 2, 3\}$ and range of $\sim = \{2, 3, 4\}$

~ is not reflexive since $(1, 1) \notin \sim$

$\therefore$ ~ is neither an equivalence relation nor a partial ordering relation.



7. Let Z be the set of all integers and N, the set of all positive integers. Find, if the relation ~ is reflexive, symmetric, antisymmetric and transitive when

(a) $x \sim y \Leftrightarrow x \leq y + 1$ in Z

(b) $x \sim y \Leftrightarrow |x - y| \leq 2$ in N

Solution

(a) $x \sim y \Leftrightarrow x \leq y + 1$ in Z

Since $x \leq x + 1$, $x \sim x$ for all x, $\therefore$ ~ is reflexive.

$x \sim y \Rightarrow x \leq y + 1 \quad \therefore y \nleq x + 1$ i.e., $y \nsim x$

$\therefore$ ~ is not symmetric.

Take $x = 0, y = 1$. Then

$0 \sim 1, 1 \sim 0$, but $0 \neq 1$ $\therefore$ ~ is not antisymmetric.

Take $x = 5, y = 4, z = 3$.

Then $x \sim y, y \sim z$, but $x \nsim z$ $(\because 5 \nleq 3 + 1)$

$\therefore$ ~ is not transitive.

(b) $x \sim y \Leftrightarrow |x - y| \le 2$ in N

$0 = |x - x| \le 2$

$\therefore x \sim x$ for each $x \in N$

$\therefore \sim$ is reflexive

$x \sim y \Rightarrow |x - y| \le 2$

$\Rightarrow |-(y - x)| \le 2$

$\Rightarrow |y - x| \le 2$

$\Rightarrow y \sim x$

$\therefore \sim$ is symmetric.

Take $x = 1, y = 2, z = 4$.

Then $x \sim y$ and $y \sim z$, but $x \sim z$

since $|x - z| = |1 - 4| = 3$.

$\therefore \sim$ is not transitive.

With $x = 1, y = 2, x \sim y, y \sim x$, but $x \ne y$

$\therefore \sim$ is not antisymmetric.

8. Let $f : Z \to Z$ be defined by,

$$f(x) = \begin{cases} \dfrac{x}{2}, & \text{if x is even} \\ \dfrac{x-1}{2}, & \text{if x is odd} \end{cases}$$

Find whether f is a bijection.

Solution

From the below diagram, f is onto, but not $1 - 1$ since 2 and 3 have the same image 1, $\therefore$ f is not a bijection.

9. Let $f : N \to N$ be such that $f(x)$ = greatest integer less than or equal to $\sqrt{x}$. Check, if f is $1 - 1$, onto.

Solution

$f(x)$ = greatest integer $\le \sqrt{x}$

$\therefore f(1) = 1, f(2) = 1 \therefore$ f is not $1 - 1$

Since all the positive integers are images, f is onto.

10. Let $Z_p = \{0, 1, 2, \ldots\ldots\ldots p - 1\}$ where p is a prime.

Define $f: Z_7 \to Z_7$ by $f(x) = 3x \pmod 7$.

Find whether f is $1 - 1$, onto. Draw also the digraph of f.

Solution

$Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$

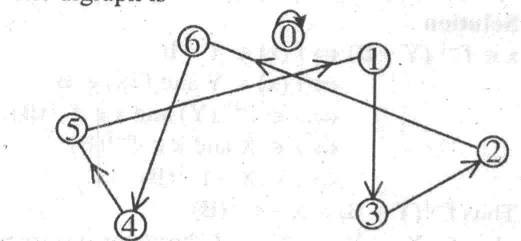
$f(0) = 0, \quad f(1) = 3$

$f(2) = 6 \quad f(3) = 2\ (\because 9 \equiv 2 \pmod 7)$

$f(4) = 5, \quad f(5) = 1, f(6) = 4$

Thus f is $1 - 1$ and onto

The digraph is



11. Let $f : R \to R$, with $f(x) = e^x$, R being the set of all real numbers. Is f, a bijective function? If f is a bijection, find its inverse.

Solution

$f(x) = f(y) \Rightarrow e^x = e^y$

$\Rightarrow e^{x-y} = 1 = e^0$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$

$\therefore$ f is $1 - 1$.

For any $y \in R$ there exists $x = \log_e y$, such that

$f(x) = e^x = e^{\log_e y} = y$.

$\therefore$ f is onto, so bijective.

Hence, $f^{-1} : R \to R$ exists and is defined by

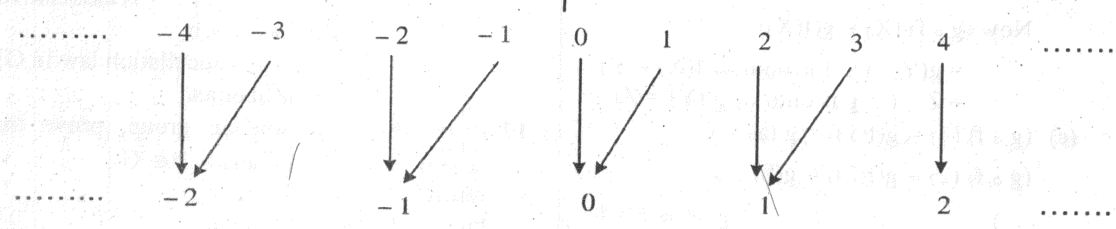
$f^{-1}(x) = \log_e x$.

12. Let $f : A \to B$, $g : B \to A$ be functions such that $f \circ g = 1_B$. Prove that f is onto.

Solution

$f \circ g : B \to B$ is such that $f \circ g = 1_B$ = identity function on B.

For any $b \in B$, $g(b) \in A$ ($\because g : B \to A$ is a function)



Let $g(b) = a$. Then,

$f \circ g = 1_B \Rightarrow (f \circ g)(b) = 1_B(b)$

$\Rightarrow f(g(b)) = b$

$\Rightarrow f(a) = b$

$\therefore$ for $b \in B$, there exists $a \in A$ such that $f(a) = b$. So f is onto.

13. Let $f : X \to Y$ and $B \subseteq Y$. Show that $f^{-1}(Y - B) = X - f^{-1}(B)$.

Solution

$x \in f^{-1}(Y - B) \Leftrightarrow f(x) \in Y - B$

$\Leftrightarrow f(x) \in Y$ and $f(x) \notin B$

$\Leftrightarrow x \in f^{-1}(Y)$ and $x \notin f^{-1}(B)$

$\Leftrightarrow x \in X$ and $x \notin f^{-1}(B)$

$\Leftrightarrow x \in X - f^{-1}(B)$

Thus $f^{-1}(Y - B) = X - f^{-1}(B)$.

14. Let $f : X \to Y$, $g : Y \to Z$. Prove or disprove the following:

(a) g is one-one $\Rightarrow g \circ f$ is one-one

(b) f, g onto $\Rightarrow (g \circ f)$ is onto

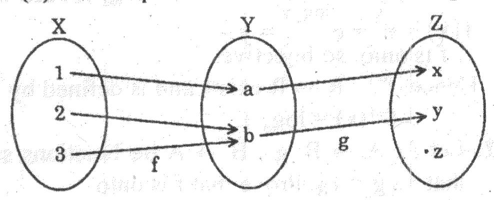
(c) $g \circ f$ is one-one $\Rightarrow$ g is one-one

(d) $g \circ f$ is onto $\Rightarrow$ f is onto

(e) $g \circ f$ is onto $\Rightarrow$ g is onto

Solution

(a) We disprove the statement. See diagram



Clearly, g is one-one. $(g \circ f)(2) = g(f(2)) = g(b) = y$;

$(g \circ f)(3) = g(f(3)) = g(b) = y \therefore g \circ f$ is not $1 - 1$.

(b) To prove $(g \circ f)(X) = Z$

Now $(g \circ f)(X) = g(f(X))$

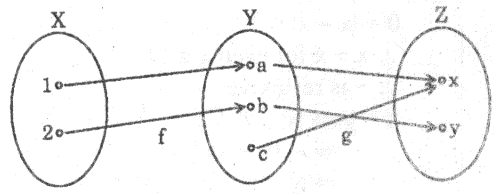
$= g(Y)$ ($\because$ f is onto $\Rightarrow f(X) = Y$)

$= Z$ ($\because$ g is onto $\Rightarrow g(Y) = Z$).

(c) $(g \circ f)(1) = g(f(1)) = g(a) = x$

$(g \circ f)(2) = g(f(2)) = g(b) = y$

$\therefore g \circ f$ is $1 - 1$



$g(a) = x = g(c) \Rightarrow$ g is not $1 - 1$.

(d) In the above diagram, $g \circ f$ is onto, but f is not onto, since c is not the image of any element in X under f.

(e) Let $z \in Z$ be arbitrary.

$g \circ f : X \to Z$ is onto $\Rightarrow$ there exists $x \in X$, such that $(g \circ f)(x) = z$

$\Rightarrow g(f(x)) = z$

$\Rightarrow g(y) = z$, where $y = f(x) \in Y$

$\therefore$ for $z \in Z$ there exists $y \in Y$ such that $g(y) = z$. So g is onto.

15. Let P(S) be the power set of a nonempty set S. Show that $(P(S), \cup)$ is a monoid.

Solution

$A, B \in P(S) \Rightarrow A \cup B \in P(S) \therefore p(S)$ is closed under $\cup$.

For $A, B, C \in P(S)$, $A \cup (B \cup C) = (A \cup B) \cup C \therefore \cup$ is associative.

$A \cup \phi = A = \phi \cup A$ for all $A \in P(S)$

$\therefore \phi \in P(S)$ is the identity element.

So $(P(S), \cup)$ is a monoid.

16. Show that in a group $(G, *)$ if for any two elements $a, b \in G$, $(a * b)^2 = a^2 * b^2$, then G is abelian.

Solution

$(a * b)^2 = a^2 * b^2 \Rightarrow (a * b) * (a * b) = (a * a) * (b * b)$

$\Rightarrow a * (b * a) * b = a * (a * b) * b$

($\because$ * is associative)

$\Rightarrow b * a = a * b$

(by cancellation law in G)

$\Rightarrow$ G is abelian.

17. If $(G, *)$ is an abelian group, prove that $(a * b)^n = a^n * b^n$ for any $a, b \in G$.

Solution

Proof is by induction on n.

When $n = 1$, $(a * b)^1 = a * b = a^1 * b^1$ $\therefore$ statement is true for $n = 1$.

Assume that $(a * b)^k = a^k * b^k$ ... (1)

Then $(a * b)^{k+1}$

$= (a * b)^k * (a * b) = (a^k * b^k) * (a * b)$ (by (1))

$= (a^k * a) * (b^k * b)$

($\because$ $*$ is commutative and associative)

$= a^{k+1} * b^{k+1}$

This shows that $(a * b)^n = a^n * b^n$ is true for $n = k + 1$.

$\therefore$ by induction, the statement is true for all n.

18. In a semi-group $(G, *)$, if $x \Delta y = x * a * y$ for a fixed element a of G, prove that Δ is an associative operation on G.

Solution

Let $x, y, z \in G$

To prove $x \Delta (y \Delta z) = (x \Delta y) \Delta z$.

Now, $x \Delta (y \Delta z) = x * a * (y \Delta z)$

$= x * a * (y * a * z)$

$= (x * a * y) * a * z$

($\because$ $*$ is associative)

$= (x \Delta y) * a * z$

$= (x \Delta y) \Delta z$.

19. Prove, by mathematical induction, the following.

(a) $1^2 + 2^2 + \dots\dots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(b) $1^3 + 2^3 + \dots\dots + n^3 = (1 + 2 + \dots\dots + n)^2$

(c) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots\dots + n(n+1)(n+2)$

$= \dfrac{1}{4} n(n+1)(n+2)(n+3)$

(d) $1 + 5 + 9 + \dots\dots + (4n - 3) = n(2n - 1)$

(e) $n < 2^n, n \geq 1$

(f) $1 + 2 + \dots\dots + n < \dfrac{1}{8}(2n+1)^2$

(g) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dots\dots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

(h) $n^3 + 2n$ is divisible by 3

(i) $\sum_{i=0}^{n-1} ar^i = \dfrac{a(1 - r^n)}{1 - r}$, $r \neq 1$

(j) $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$ = where $\overline{A_i}$ = complement of A_i and $A_1, A_2, \dots\dots, A_n$ are subsets of a universal set $\cup$.

Solution

(a) Let $P(n)$: $1^2 + 2^2 + \dots\dots + n^2$

$= \dfrac{1}{6} n(n+1)(2n+1)$

When $n = 1$, $\dfrac{1}{6} n(n+1)(2n+1)$

$= \dfrac{1}{6} \times 1 \times 2 \times 3 = 1 = 1^2$

$\therefore$ P(1) is true.

Assume that P(k) is true.

i.e., $1^2 + 2^2 + \dots + k^2 = \dfrac{1}{6} k(k+1)(2k+1)$ ... (1)

(Induction hypothesis)

$\therefore$ $1^2 + 2^2 + \dots\dots + (k+1)^2$

$= [1^2 + 2^2 + \dots\dots + k^2] + (k+1)^2$

$= \dfrac{1}{6} k(k+1)(2k+1) + (k+1)^2$ (by (1))

$= \dfrac{(k+1)}{6} [k(2k+1) + 6(k+1)]$

$= \dfrac{(k+1)}{6} [2k^2 + 7k + 6]$

$= \dfrac{1}{6} (k+1)(k+2)(2k+3)$

$= \dfrac{1}{6} (k+1)(k+1+1)(2(k+1)+1)$

This shows that P (k + 1) is also true. $\therefore$ P(n) is true for all n.

(b) Let $P(n)$: $1^3 + 2^3 + \dots\dots + n^3$

$= (1 + 2 + \dots\dots + n)^2$

When $n = 1$, left hand side $= 1^3 = 1 = 1^2 =$ right hand side

$\therefore$ P(1) is true.

Assume that P(k) is true.

i.e., $1^3 + 2^3 + \dots\dots + k^3 = (1 + 2 + \dots\dots + k)^2$ ... (1)

Then $1^3 + 2^3 + \dots\dots + (k+1)^3$

$= [1^3 + 2^3 + \dots\dots + k^3] + (k+1)^3$

$= (1 + 2 + \dots\dots + k)^2 + (k+1)^3$ (by (1))

$= \left[\dfrac{k(k+1)}{2}\right]^2 + (k+1)^3$

$$(\because 1 + 2 + \cdots\cdots + n = \frac{n(n+1)}{2})$$

$$= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)]$$

$$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2$$

$$= [1 + 2 + \cdots\cdots + (k+1)]^2$$

Thus P(k + 1) is true. $\therefore$ P(n) is true for all n.

(c) Let P(n): $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots\cdots + n(n+1)(n+2) = \frac{1}{4} n(n+1)(n+2)(n+3)$

When n = 1, left hand side = $1 \cdot 2 \cdot 3 = 6$

and right hand side = $\frac{1}{4} \times 1 \times 2 \times 3 \times 4 = 6$

$\therefore$ P(1) is true.

Assume that P(k) is true.

That is,

$$1{\cdot}2{\cdot}3 + 2{\cdot}3{\cdot}4 + \cdots\cdots + k(k+1)(k+2) = \frac{1}{4}k(k+1)(k+2)(k+3) \quad \ldots (1)$$

Now, $1{\cdot}2{\cdot}3 + 2{\cdot}3{\cdot}4 + \cdots\cdots + (k+1)(k+2)(k+3)$

$$= 1 \cdot 2 \cdot 3 + \cdots\cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$$

$$= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3) \quad \text{(by (1))}$$

$$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$$

This shows that P(n) is true for n = k + 1. $\therefore$ P(n) is true for all n.

(d) Let P(n) be the statement

$$1 + 5 + 9 + \cdots\cdots + (4n-3) = n(2n-1).$$

When n = 1, left hand side = 1 and right hand side = $1(2 \times 1 - 1) = 1$, so that P (1) is true.

Assume that P (k) is true.

Then $1 + 5 + 9 + \cdots\cdots + (4k-3) = k(2k-1)$ ... (1)

Now, $1 + 5 + 9 + \cdots\cdots + [4(k+1) - 3]$

$$= 1 + 5 + \cdots\cdots + (4k-3) + (4k+1)$$

$$= k(2k-1) + (4k+1) \quad \text{(by (1))}$$

$$= 2k^2 + 3k + 1$$

$$= (k+1)(2k+1)$$

$$= (k+1)[2(k+1) - 1]$$

Thus, P(k + 1) is true and so P(n) is true for all n.

(e) Let P(n) : $n < 2^n$, $n \geq 1$.

Then P(1) is true since $1 < 2^1$

Assume that P(k) is true. Then $k < 2^k$ ...(1)

Now $k + 1 < 2^k + 1$ (by (1))

$$< 2^k + 2^k \quad (\because 1 < 2^k \text{ always})$$

$$= 2^k \times 2$$

$$= 2^{k+1}$$

$\therefore$ P(k + 1) is true and so P(n) is true for $n \geq 1$.

(f) P(n) : $1 + 2 + \ldots\ldots + n < \frac{1}{8}(2n+1)^2$

When n = 1, $\frac{1}{8}(2n+1)^2 = \frac{9}{8}$ and $1 < \frac{9}{8}$

$\therefore$ P(1) is true.

Assume that P(k) is true. Then

$$1 + 2 + \cdots\cdots + k < \frac{1}{8}(2k+1)^2 \quad \ldots (1)$$

Now $1 + 2 + \cdots\cdots + (k+1) = (1 + 2 + \cdots\cdots + k) + (k+1)$

$$< \frac{1}{8}(2k+1)^2 + (k+1) \quad \text{(by (1))}$$

$$= \frac{1}{8}[4k^2 + 4k + 1 + 8(k+1)]$$

$$= \frac{1}{8}(4k^2 + 12k + 9)$$

$$= \frac{1}{8}(2k+3)^2$$

$$= \frac{1}{8}[2(k+1) + 1]^2.$$

$\therefore$ P(k + 1) is true. Thus P(n) is true for all n.

(g) P(n) : $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots\cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

When n = 1, L.H.S. (left-hand side) =

$\frac{1}{1 \cdot 2} = \frac{1}{2}$ = R.H.S (right-hand side)

$\therefore$ P(1) is true.

Assuming P(k) to be true, we have

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad \ldots(1)$$

Now

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\cdots+\frac{1}{(k+1)(k+2)}$$

$$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad \text{(by(1))}$$

$$=\frac{1}{(k+1)(k+2)}[k(k+2)+1]$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{k+2}$$

$$=\frac{k+1}{(k+1)+1}$$

Hence, P(k + 1) is true. Thus P(n) is true for all n.

(h) P(n): n^3+2n is divisible by 3.

When n = 1, $n^3+2n=3$ which is divisible by 3.

$\therefore$ P(1) is true.

Assume that P(k) is true. That is, k^3+2k is divisible by 3 ... (1)

Now $(k+1)^3+2(k+1)=(k^3+2k)+3(k^2+k+1)$

By (1), k^3+2k is divisible by 3; also $3(k^2+k+1)$ is divisible by 3.

$\therefore$ $(k+1)^3+2(k+1)$ is divisible by 3. That is, P(k + 1) is true. So P(n) is true for all n.

(i) $\sum_{i=0}^{n-1} ar^i = a+ar+ar^2+\cdots\cdots+ar^{n-1}$

Let P(n): $a+ar+ar^2+\cdots\cdots+ar^{n-1}=\frac{a(1-r^n)}{1-r}$, $r\neq 1$.

When n = 1, R.H.S $=\frac{a(1-r^n)}{1-r}=\frac{a(1-r)}{1-r}$

$= a =$ L.H.S

$\therefore$ P(1) is true.

Assume that P(k) is true.

Then $a+ar+\cdots\cdots+ar^{k-1}=\frac{a(-r^k)}{1-r}$ ...(1)

Now, $a+ar+\cdots\cdots+ar^k=(a+ar+\ldots\ldots+ar^{k-1})+ar^k$

$$=\frac{a(1-r^k)}{1-r}+ar^k \text{ (by (1))}$$

$$=\frac{a-ar^k+ar^k-ar^{k+1}}{1-r}$$

$$=\frac{a(1-r^{k+1})}{1-r}$$

Thus, P (k + 1) is true. $\therefore$ P(n) is true for all n.

(j) P(n): $\overline{A_1\cap A_2\cap \cdots\cdots\cap A_n}$

$=\overline{A_1}\cup\overline{A_2}\cup\cdots\cdots\cup\overline{A_n}$

Clearly P(1) is true, since both sides are $\overline{A_1}$.

Assuming P(k) to be true, we have

$$\overline{A_1\cap A_2\cap\cdots\cdots\cap A_k}$$

$$=\overline{A_1}\cup\overline{A_2}\cup\cdots\cdots\cup\overline{A_k} \quad \ldots(1)$$

Now, $\overline{A_1\cap A_2\cap\cdots\cdots\cap A_{k+1}}$

$$=\overline{(A_1\cap A_2\cap\cdots\cdots\cap A_k)\cap A_{k+1}}$$

$$=\overline{A\cap A_{k+1}}, \quad \ldots(2)$$

where $A=A_1\cap A_2\cap\cdots\cdots\cap A_k$

$=\overline{A}\cup\overline{A_{k+1}}$ (by De Morgan's law)

$= \overline{A_1 \cap A_2 \cap \ldots\ldots \cap A_k} \cup \overline{A_{k+1}}$

$= \overline{A_1} \cup \overline{A_2} \cup \ldots\ldots \cup \overline{A_k} \cup \overline{A_{k+1}}$ (by (1))

Thus, P(k + 1) is true and hence P(n) is true for all n.

3.8 RECURRENCE RELATIONS

Definition 1: Let $a_0, a_1, a_2, \ldots\ldots a_n, \ldots\ldots$ be a sequence of numbers. An equation relating a number a_n, for any n, to some of its predecessors $a_0, a_1, \ldots\ldots a_{n-1}$ is called a **recurrence relation** or **difference equation**.

Example: Consider the geometric series

$$1, 5, 5^2, \ldots\ldots, 5^n, \ldots\ldots\ldots\ldots$$

Here $a_0 = 1, a_1 = 5, a_2 = 5^2 \ldots\ldots\ldots$ so that $a_n = 5^n$

Note that $a_n = 5^n = 5 \cdot 5^{n-1} = 5a_{n-1}$

Thus $a_n = 5a_{n-1}$ with $a_0 = 1$ is the required recurrence relation.

Example: We obtain a recurrence relation for the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ... as follows:

Here $a_0 = 1$, $a_1 = 1$, $a_2 = 1 + 1 = a_0 + a_1$, $a_3 = 3 = 1 + 2 = a_1 + a_2$, $a_4 = 5 = a_2 + a_3 \ldots\ldots\ldots$,

So that $a_n = a_{n-2} + a_{n-1}$.

Thus $a_n = a_{n-1} + a_{n-2}$ with $a_0 = a_1 = 1$ is the required Fibonacci sequence.

Example: Suppose there are n ovals drawn on the plane. Assume that (i) an oval intersects each of the other ovals at exactly two points and (ii) no three ovals meet at the same point. The problem is to obtain a recurrence relation which tells the number of regions into which the plane is divided by these ovals.

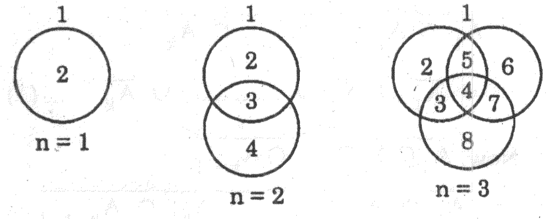


Fig.

Let a_n = number of regions determined by the n oval clearly $a_1 = 2$, $a_2 = 4$, $a_3 = 8$ (see **Figure**).

Assume that we have drawn n – 1 ovals which divide the plane into a_{n-1} regions. Then the nth oval will intersect these n – 1 ovals at exactly 2(n – 1) points. That is, the nth oval will be divided into 2(n – 1) arcs. Each of these arcs will divide one of the a_{n-1} regions into two regions.

$\therefore a_n = a_{n-1} + 2(n-1)$ with $a_1 = 2$ is the required recurrence relation.

Remark: Suppose a recurrence relation is given. To initiate the computation, one or several numbers in the sequence $a_0, a_1, a_2 \ldots\ldots$ must be known. These initial values are known as **boundary condition**.

In example 1, boundary condition is $a_0 = 1$; in the case of example 2, it is $a_0 = 1, a_1 = 1$.

Definition 2: A recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + \ldots\ldots + c_r a_{n-r} = f(n) \quad \ldots (1)$$

where c_i's are constants is called **a linear recurrence relation with constant coefficients.**

Solution of the recurrence relation (1) is obtained as follows.

Any solution satisfying

$$c_0 a_n + c_1 a_{n-1} + \ldots\ldots + c_r a_{n-r} = 0 \quad \ldots (2)$$

is known as the **homogeneous solution** of (1).

It is denoted by $a_n^{(h)}$. Any solution satisfying (1) is called the **particular solution** of (1). It is denoted by $a_n^{(p)}$. Thus the **general solution** of (1) is $a_n^{(h)} + a_n^{(p)}$

Procedure to obtain the general solution of (1)

(i) Let $a_n = A\alpha^n$ be a trial solution of the equation (2). Then we get an equation of the form

$$c_0\alpha^r + c_1\alpha^{r-1} + \ldots\ldots + c_{r-1}\alpha + c_r = 0 \quad \ldots (3)$$

Equation (3) is known as the **characteristic equation** of the given equation (1). Its roots are called the characteristic roots of the equation (1). Equation (3), being an equation of degree r has r roots, say, $\alpha_1, \alpha_2, \ldots\ldots \alpha_r$.

(ii) (a) If all the roots are distinct and real, then

$$a_n^{(h)} = A_1\alpha_1^n + A_2\alpha_2^n + \dots\dots\dots + A_r\alpha_r^n,$$

where A_i's are constants.

(b) If α_1 is a multiple real root of order k, then $\left(A_1 n^{k-1} + A_2 n^{k-2} + \dots + A_k\right)\alpha_1^n$ is the corresponding homogeneous part.

(c) If $\alpha_1 = \alpha + i\beta$ is a complex root, then $\alpha_2 = \alpha - i\beta$ is also a root of the characteristic equation.

Let $\phi^2 = \alpha^2 + \beta^2$, $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

Then $B_1\rho^n \cos n\theta + B_2\rho^n \sin n\theta$ is the corresponding part of the homogeneous solution wherever $\alpha + i\beta$ is a complex root of the equation, (β_1, β_2 are constants).

Remark: There is no general method to obtain the particular solution $a_n^{(p)}$ of the given equation (1).

Example: Solve the recurrence relation

$$a_n = a_{n-1} + a_{n-2},\ a_0 = 1,\ a_2 = 1.$$

We have $a_n - a_{n-1} - a_{n-2} = 0$ ... (1)

$\therefore$ the characteristic equation is $\alpha^2 - \alpha - 1 = 0$

$$\therefore \alpha = \frac{1 \pm \sqrt{5}}{2}$$

Put $\alpha = \frac{1+\sqrt{5}}{2}$, $\alpha_2 = \frac{1-\sqrt{5}}{2}$

Then, $a_n^{(h)} = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n$

$a_n^{(p)} = 0$ since right hand side of (1) is 0

$\therefore$ general solution is a_n

$$= A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n A_2 \quad \dots (2)$$

Since $a_0 = 1$, by (2), $A_1 + A_2 = 1$ and $a_1 = 1$, by (2) gives

$$A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Solving, $A_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}$, $A_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$

Using these in (2),

$$a_n = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Example: Solve $a_n - 2a_{n-1} + a_{n-2} = 0$, $a_1 = 2$, $a_2 = 3$.

The characteristic equation is $\alpha^2 - 2\alpha + 1 = 0$

$\therefore a_n^{(h)} = A_1 n + A_2$ and $a_n^{(p)} = 0$.

$\therefore a_n = A_1 n + A_2$.

$a_1 = 2 \Rightarrow 2 = A_1 + A_2$

$a_2 = 3 \Rightarrow 3 = 2A_1 + A_2$

Solving $A_1 = 1$, $A_2 = 1$

$\therefore$ solution is $a_n = n + 1$.

Example: Solve $a_n = a_{n-1} - a_{n-2}$, $a_1 = 1$, $a_2 = 0$.

The characteristic equation $\alpha^2 - \alpha + 1 = 0$ has the roots

$\alpha_1 = \frac{1 + i\sqrt{3}}{2}$, $\alpha_2 = \frac{1 - i\sqrt{3}}{2}$ which are complex.

$\therefore a_n^{(h)} = B_1\rho^n \cos n\theta + B_2\rho^n \sin n\theta$ where B_1, B_2 are constants, $\rho^2 = \frac{1}{4} + \frac{3}{4} = 1$, $\theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

$$\therefore a_n = a_n^{(h)} = B_1 \cos\frac{n\pi}{3} + B_2 \sin\frac{n\pi}{3}.$$

$$\left(\text{since } a_n^{(p)} = 0\right)$$

$$a_1 = 1 \Rightarrow 1 = B_1 \cos\frac{\pi}{3} + B_2 \sin\frac{\pi}{3} = \frac{1}{2}\left(B_1 + \sqrt{3}\, B_2\right)$$

$$a_2 = 0 \Rightarrow 0 = B_1 \cos\frac{2\pi}{3} + B_2 \sin\frac{2\pi}{3}$$

$$= \frac{1}{2}\left(-B_1 + \sqrt{3}\, B_2\right)$$

Solving $B_1 = 1, B_2 = \frac{1}{\sqrt{3}}$

$$\therefore a_n = \cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{n\pi}{3}$$

Example: Solve $a_n + 2a_{n-1} = n + 3$, $a_0 = 3$. .... (1)

In this case, the characteristic equation is $\alpha + 2 = 0$.

$\therefore \alpha = -2$

$$a_n^{(h)} = A(-2)^n$$

Assume that Bn + c is the particular solution of the equation. Then from (1),

$(Bn + c) + 2\,[B(n-1) + c] = n + 3$ $(\because a_n = Bn + c)$

Equating coefficients of n and constant terms

$3B = 1, 3C - 2B = 3$

$$\therefore B = \frac{1}{3}, C = \frac{11}{9}$$

Thus $a_n^{(p)} = Bn + c = \frac{n}{3} + \frac{11}{9}$

$\therefore$ general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$= A(-2)^n + \frac{n}{3} + \frac{11}{9}$$

Since $a_0 = 3$, $3 = A + \frac{11}{9}$ $\therefore A = \frac{16}{9}$

$$\therefore a_n = \frac{16}{9}(-2)^n + \frac{n}{3} + \frac{11}{9}.$$

Example: Solve $a_n + 2a_{n-1} + a_{n-2} = 2^n$.

Since $\alpha^2 + 2\alpha + 1 = 0$, $\alpha = -1, -1$

$\therefore a_n^{(h)} = (A_1 + A_2 n)(-1)^n$

Assume $A \cdot 2^n$ as the particular solution. Then

$$A\cdot 2^n + 2(A\cdot 2^{n-1}) + A\cdot 2^{n-2} = 2^n$$

or $2A + \frac{A}{4} = 1$ $\therefore A = \frac{4}{9}$

$\therefore a_n^{(p)} = \frac{4}{9}2^n$. So general solution is

$$a_n = (A_1 + A_2 n)(-1)^n + \frac{4}{9}\cdot 2^n$$

where A_1, A_2 are constants which can be found by the boundary conditions.

OBJECTIVE QUESTIONS

1. Solution of the recurrence relation $a_n = 2a_{n-1} + 4^{n-1}$, $a_1 = 3$ is

   (a) $a_n = \frac{2^n}{2}$

   (b) $a_n = \frac{2^n}{2} + 4^{n-1}$

   (c) $a_n = \frac{2^n}{2} + \frac{4^n}{2}$

   (d) none of the above

2. Let $\{f_n\}$ and $\{g_n\}$ be two sequences of numbers given by the recurrence relations $f_{n+1} = f_n + g_n$, $g_{n+1} = -f_n + g_n$. Then f_{4n} and g_{4n} are given by

   (a) $f_{4n} = 2^{2n} f_o$, $g_{4n} = 2^{2n} g_o$

   (b) $f_{4n} = 2^{2n} g_o$, $g_{4n} = 2^{2n} f_o$

   (c) $f_{4n} = -2^{2n} g_o$, $g_{4n} = -2^{2n} f_o$

   (d) $f_{4n} = (-1)^n 2^{2n} f_o$, $g_{4n} = (-1)^n 2^{2n} g_o$

Directions

Consider the following function

Function (n, m: integer): integer;

begin

If (n ⇐ 0) or (m ⇐ 0) then F : = 1

else

F : = F (n – 1, m) + F (n, m – 1)

end;

Use the recurrence relations $^nC_R = {}^{n-1}C_R + {}^{n-1}C_{R-1}$ to answer the following **two** questions. Assume that n, m are positive integers.

3. The value of F (n, 2) is

   (a) $^{n+2}C_1$ **(b)** $^{n+2}C_2$

   (c) $^{n+1}C_2$ **(d)** $^{n+1}C_1$

4. The value of F (n, m) is

(a) ${}^{n+m}C_m$ (b) ${}^{n+m}C_{m-1}$

(c) ${}^{n+m}C_n$ (d) ${}^{n+m}C_{n-1}$

Directions:

Let F be the set of 1–1 functions from the set $\{1, 2, ..., n\}$ to the set $\{1, 2, ..., m\}$ where $m \geq n \geq 1$. Answer the following **two** questions:

5. How many functions are members of F?

(a) ${}^{n}P_m$ (b) ${}^{m}P_n$

(c) ${}^{m}P_{n-1}$ (d) ${}^{m-1}P_n$

6. How many functions f in F have the property $f(i) = 1$ for some $1 \leq i \leq n$?

(a) $m(m-1)(m-2)...(m-n)$

(b) $m(m-1)(m-2)...(m-n-1)$

(c) $(m-1)(m-2)...(m-n+1)$

(d) none of these

KEY

1. (c)	**2.** (d)	**3.** (b)	**4.** (a)
5. (b)	**6.** (c).		

SELECTED QUESTIONS AND ANSWERS

GATE 2002

1. "If X then Y unless Z" is represented by which of the following formulas in prepositional logic? ("$\neg$" is negation, "$\wedge$" is conjunction, and "$\rightarrow$" is implication)

(A) $(X \wedge \neg Z) \rightarrow Y$

(B) $(X \wedge Y) \rightarrow \neg Z$

(C) $X \rightarrow (Y \wedge \neg Z)$

(D) $(X \rightarrow Y) \wedge \neg Z$

Ans: (A)

2. Four fair coins are tossed simultaneously. The probability that at least one head and one tail turn up is

(A) 1/16 (B) 1/8

(C) 7/8 (D) 15/16

Ans: (C)

3. The binary relation $S = \phi$ (empty set) on set $A = \{1, 2, 3\}$ is

(A) neither reflexive nor symmetric.

(B) symmetric and reflexive.

(C) transitive and reflexive.

(D) transitive and symmetric.

Ans: (B)